How to... Compute Eigenvalues and -vectors

Given: A quadratic matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Wanted: Eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ such that

$$\mathbf{A}\,\boldsymbol{v}_i = \lambda_i\,\boldsymbol{v}_i \qquad i = 1, \ldots, n$$

and algebraic and geometric multiplicities $\mu_{\mathbf{A}}(\lambda_i)$ and $\gamma_{\mathbf{A}}(\lambda_i)$.

Example

We consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

1 Computation of the characteristic polynomial

Compute the matrix $\mathbf{A} - \lambda \cdot \mathbf{I}_n$ (i.e. subtract λ from every diagonal element of $\mathbf{A}$). Then compute the *characteristic polynomial* $P_{\mathbf{A}}(\lambda)$ of $\mathbf{A}$, that is the determinant of the matrix set up before, i.e.,

$$P_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \cdot \mathbf{I}_n).$$

The result is a polynomial in the variable λ.

First, we compute the matrix

$$\mathbf{A} - \lambda \cdot \mathbf{I}_3 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 0 \\ -2 & 2 & 3-\lambda \end{pmatrix}$$

and then the characteristic polynomial

$$\begin{aligned} P_{\mathbf{A}}(\lambda) &= \det \begin{pmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 0 \\ -2 & 2 & 3-\lambda \end{pmatrix} \\ &= (3-\lambda) \cdot \det \begin{pmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda)^2. \end{aligned}$$

2 Computation of the eigenvalues

Compute the roots of the characteristic polynomial, i.e., solve

$$0 \overset{!}{=} P_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \cdot \mathbf{I}_n)$$

for λ. This equation has n complex solutions (but maybe less than n real solutions).

We want to compute the roots of $P_{\mathbf{A}}(\lambda)$. Thus, we solve

$$(3 - \lambda)(1 - \lambda)^2 = 0.$$

This equation has the solutions $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = 1$. Thus, the matrix $\mathbf{A}$ has the eigenvalues 1 and 3.

3 Computation of the eigenvectors

For every (distinct) eigenvalue λ_i solve the system of linear equations

$$(\mathbf{A} - \lambda_i \mathbf{I}_n)\,\boldsymbol{v} = 0.$$

This system must have infinitely many solutions. The set of all solutions to this system of equations is called *eigenspace* of the eigenvalue λ_i. Any vector from this solution set (except for the zero vector) is an *eigenvalue*.

We want to compute the eigenspaces of the eigenvalues $\lambda = 1$ and $\lambda = 3$.
For $\lambda = 1$ need to solve

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution to this system of linear equations is

$$\mathcal{L}_{\lambda=1} = \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

Thus the set $\mathcal{L}_{\lambda=1}$ is the eigenspace of $\lambda = 1$, and every vector $\alpha \cdot (1, 0, 1)^\top$ with $\alpha \neq 0$ is an eigenvector for $\lambda = 1$.

For $\lambda = 3$ need to solve

$$\begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution to this system of linear equations is

$$\mathcal{L}_{\lambda=3} = \left\{ \alpha \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

Thus the set $\mathcal{L}_{\lambda=3}$ is the eigenspace of $\lambda = 3$, and every vector $\alpha \cdot (0, 0, 1)^\top$ with $\alpha \neq 0$ is an eigenvector for $\lambda = 3$.

4 Algebraic and geometric multiplicities

For every eigenvalue λ_i determine the multiplicity of this root in $P_\mathbf{A}(\lambda)$, i.e. count how often this value appears in the solution of $P_\mathbf{A}(\lambda) = 0$. This number is the *algebraic multiplicity* $\mu_\mathbf{A}(\lambda_i)$ of λ_i.
The dimension (the maximal number of linear independent vectors) in the eigenspace of λ_i is the *geometric multiplicity* $\gamma_\mathbf{A}(\lambda_i)$ of the eigenvalue λ_i.

As it can be seen in step 2, the root $\lambda = 3$ appears once in $P_\mathbf{A}(\lambda)$ and the root $\lambda = 1$ appears twice in $P_\mathbf{A}(\lambda)$. Thus the algebraic multiplicities are

$$\mu_\mathbf{A}(3) = 1 \qquad \mu_\mathbf{A}(1) = 2.$$

Both eigenspaces are just sets of multiples of a vector, thus the dimension of both eigenspaces is 1. This means the geometric multiplicities are

$$\gamma_\mathbf{A}(3) = 1 \qquad \gamma_\mathbf{A}(1) = 1.$$