Vectors and Matrices

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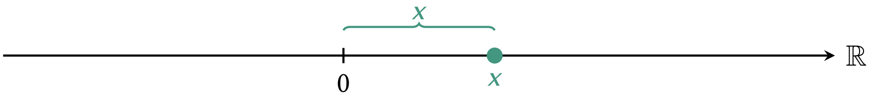
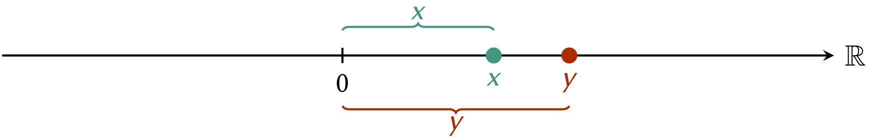
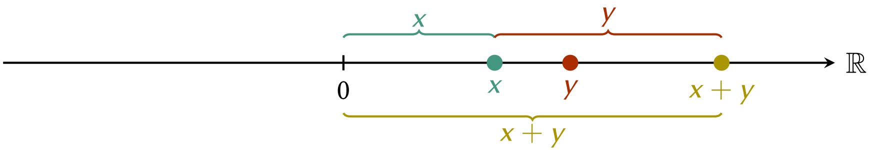
Real numbers



Real numbers



Real numbers



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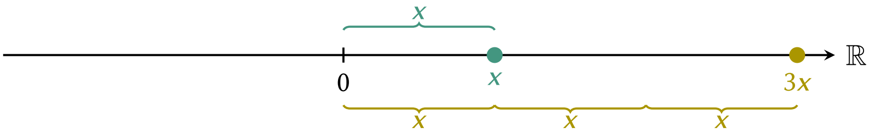

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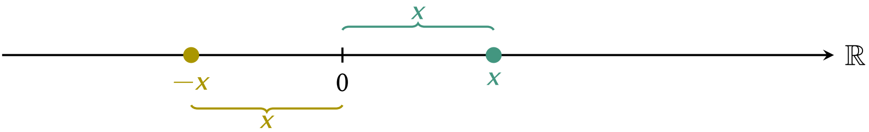
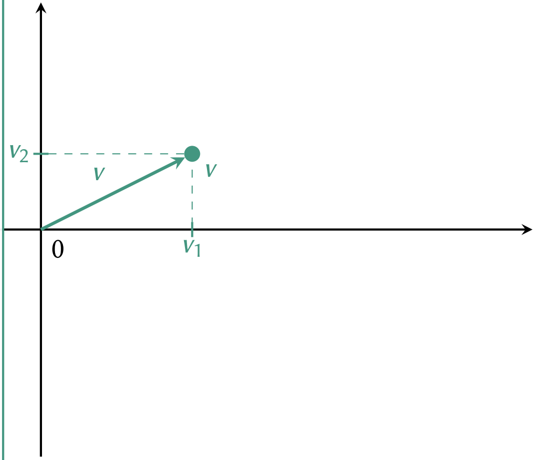


Vectors

Vectors

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$$

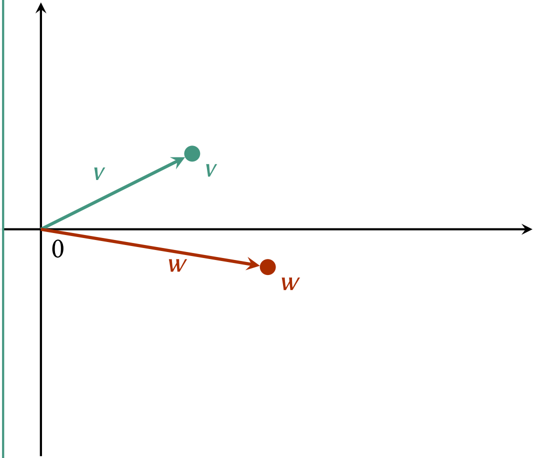
An ordered n-tupel
of real numbers
is called **vector** in $\mathbb{R}^n$



Vectors

Vectors

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix}$$



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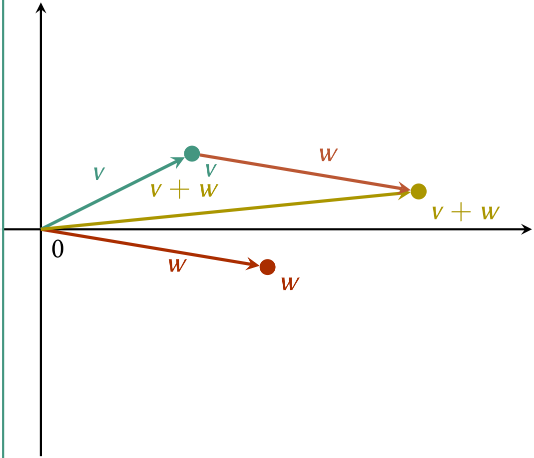
Addition

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_{n-1} + w_{n-1} \\ v_n + w_n \end{bmatrix}$$



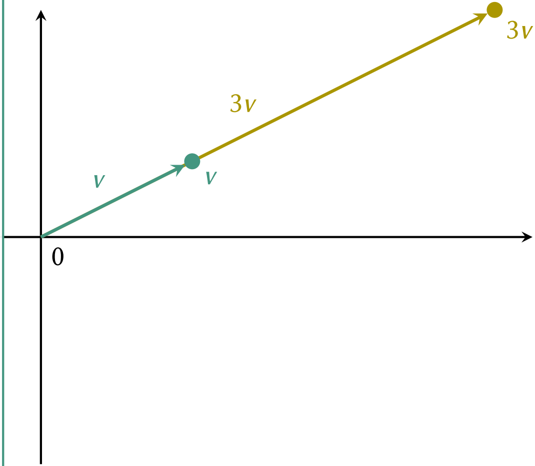
Vectors

Vectors

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$$

Scalar Multiplication

$$\lambda \boldsymbol{v} = \begin{bmatrix} \lambda\, v_1 \\ \lambda\, v_2 \\ \vdots \\ \lambda\, v_{n-1} \\ \lambda\, v_n \end{bmatrix}$$

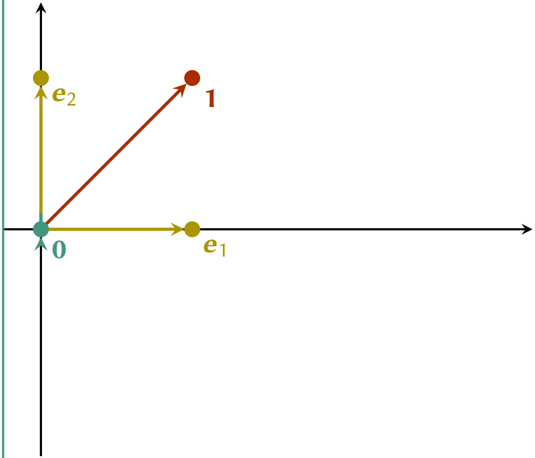


Vectors

Special Vectors

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ *i-th unit vector*



Vectors

Dot Product

$$\langle v, w \rangle := \sum_{k=1}^{n} v_k \cdot w_k$$

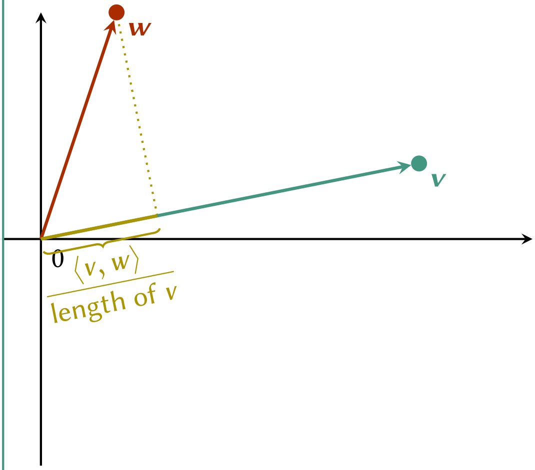
Example:

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\rangle = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$$

Fact:

v and w are orthogonal

$\Leftrightarrow \langle v, w \rangle = 0$

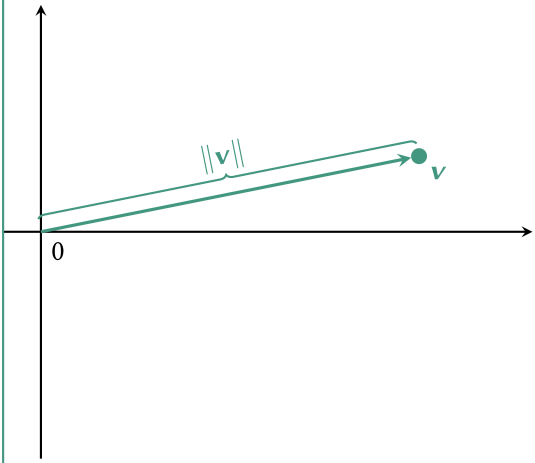


Vectors

(Euclidean) Norm

$$\|v\| := \sqrt{\langle v, v \rangle}$$
$$= \sqrt{\sum_{k=1}^{n} v_k \cdot w_k}$$

Length of the vector



- A $n \times m$ **matrix** is a rectangular array of real numbers with n rows and m columns

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Note: Vectors are in particular matrices with one column

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Note: Vectors are in particular matrices with one column

- **Addition of matricies**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$$

Scalar multiplication

Let $\lambda \in \mathbb{R}$.

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 & 1 \cdot 8 + 2 \cdot 10 \\ 3 \cdot 7 + 4 \cdot 9 & 3 \cdot 8 + 4 \cdot 10 \\ 5 \cdot 7 + 6 \cdot 9 & 5 \cdot 8 + 6 \cdot 10 \end{pmatrix}$$

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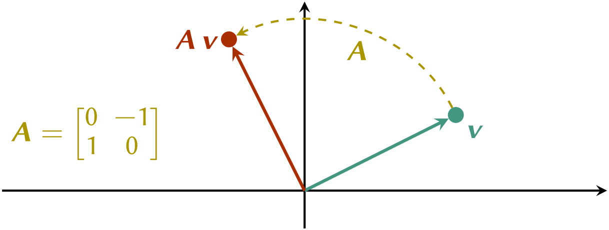
- Every matrix $\boldsymbol{A}$ defines a **linear function** $f_{\boldsymbol{A}} : \mathbb{R}^m \to \mathbb{R}^n$ with $f_{\boldsymbol{A}}(\boldsymbol{v}) = \boldsymbol{A}\,\boldsymbol{v}$

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Zero-Matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Diagonal-Matrix

$$\operatorname{diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Identity-Matrix

$$I_n = \operatorname{diag}(1, \ldots, 1) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Transpose of a Matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \qquad \boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

Transpose of a Matrix

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Example:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^T = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix}$$