IRTG 1792 Discussion Paper 2018-010

How Sensitive are Tail-related Risk Measures in a Contamination Neighbourhood?

Wolfgang Karl Härdle *
Chengxiu Ling *[2]

* Humboldt-Universität zu Berlin, Germany
*[2] University of Lausanne, Switzerland

This research was supported by the Deutsche Forschungsgemeinschaft through the International Research Training Group 1792 "High Dimensional Nonstationary Time Series".

http://irtg1792.hu-berlin.de
ISSN 2568-5619

International Research Training Group 1792

How Sensitive are Tail-related Risk Measures in a Contamination Neighbourhood?*

Wolfgang Karl Härdle[†], Chengxiu Ling[‡]

Abstract

Estimation or mis-specification errors in the portfolio loss distribution can have a considerable impact on risk measures. This paper investigates the sensitivity of tail-related risk measures including the Value-at-Risk, expected shortfall and the expectile-quantile transformation level in an epsilon-contamination neighbourhood. The findings give the different approximations via the tail heaviness of the contamination models and its contamination levels. Illustrating examples and an empirical study on the dynamic CRIX capturing and displaying the market movements are given. The codes used to obtain the results in this paper are available via https://github.com/QuantLet/SRMC .

JEL classification: C13, G10, G31

Keywords: Sensitivity, expected shortfall, expectile, Value-at-Risk, risk management, influence function, CRIX

*Financial support from the Deutsche Forschungsgemeinschaft via IRTG 1792 "High Dimensional Non Stationary Times Series", Humboldt-Universität zu Berlin, the National Natural Science Foundation of China grant (11604375), and Chinese Government Scholarship (201708505031), are gratefully acknowledged.

†Humboldt-Universität zu Berlin, C.A.S.E. - Center for Applied Statistics and Economics, Spandauer Str. 1, 10178 Berlin, Germany and SKBI School of Business, Singapore Management University, 50 Stamford Road, Singapore 178899, tel. +49 (0)30 2093-5630. E-Mail: haerdle@hu-berlin.de

‡School of Mathematics and Statistics, 400715 Chongqing, China and Department of Actuarial Science, University of Lausanne, Chamberonne 1015 Lausanne, Switzerland, E-Mail: chengxiu.ling@unil.ch

1 Introduction

Value-at-Risk (VaR) measures the probability of tail events of a given portfolio over a prescribed holding period. Specifically, the VaR of $X \sim F$ at probability level $\alpha \in (0, 1)$ is given as

$$q_\alpha = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\},$$

i.e., the α quantile q_α of the cumulative distribution function (cdf) F of the underlying risk X. Although VaR has become some sort of standard measure of financial market risk, it has been criticized for reporting only a tail probability, and thus neglecting effects like the amount of loss beyond the quantile. Additionally, VaR does not take diversification and risk aggregation effects into account. Expected shortfall (ES), a natural coherent alternative to VaR, overcomes these weaknesses, is becoming increasingly used in financial risk management, Artzner et al. (1999); Delbaen (2002). Specifically, ES is defined as the conditional expectation of the loss given that it is smaller than VaR, i.e.,

$$ES_\alpha = \mathsf{E}\left[X | X < q_\alpha\right] \tag{1.1}$$

provided that the underlying distribution function F is continuous. Cont et al. (2010) pointed out that ES appears to lack robustness with respect to small changes in the underlying cdf. The recent contribution by Mihoci et al. (2017) provides evidence on expected shortfall robustness through its link with expectile, which is given by minimizing the asymmetric weighted least square error, Newey and Powell (1987)

$$e_\alpha = \arg\min_\theta \mathsf{E}\left[|\alpha - \mathbf{I}\{X - \theta < 0\}||X - \theta|^2\right],$$

where $\mathbf{I}\{\cdot\}$ stands for the indicator function. It is well-known that the expectile is the only coherent risk measure possessing elicitability, a desirable property for model selection, generalized regression, forecast ranking and comparative backtesting, Nolde et al. (2017); Xu et al. (2018). Further, the expectile is the so-called index of prudentiality in financial set-up, i.e., the amount of money added to a position with a pre-specified, sufficiently high gain-loss ratio, Bellini and Di Bernardino (2017);

Daouia et al. (2017).

To find the expectile-quantile transformation level is practically useful for the regulators to set a proper level of quantile with the extreme loss taken into account, Kuan et al. (2009); Borke and W.K. (2018). Specifically, using Jones (1994) it is not hard to verify the expectile is obtainable through a one-to-one mapping with VaR. In other words, if $e_{w_\alpha} = q_\alpha$ for some α given, then the corresponding level w_α is such that

$$w_\alpha = \frac{LPM_\alpha - q_\alpha \alpha}{2(LPM_\alpha - q_\alpha \alpha) + q_\alpha - \mathsf{E}[X]}, \tag{1.2}$$

Yao and Tong (1996). Here $LPM_\alpha = \int_{-\infty}^{q_\alpha} x \, \mathrm{d}F(x)$ stands for the lower partial moment at α quantile. As a consequence, we get an alternative expression of ES as follows.

$$ES_\alpha = q_\alpha + \frac{q_\alpha - \mathsf{E}[X]}{1 - 2w_\alpha} \frac{w_\alpha}{\alpha}, \tag{1.3}$$

Taylor (2008); Mihoci et al. (2017).

Financial asset returns and fundamental factor exposure data often contain outliers, observations that are inconsistent with the majority of the data. One might then be interested in the contamination case as follows.

$$F_\epsilon(x) = (1 - \epsilon)F(x) + \epsilon H(x), \quad x \in \mathbb{R}, \ \epsilon \in [0, 1], \tag{1.4}$$

where ϵ reflects the amount of uncertainty in F, and H represents plausible deviations from F. Note that (1.4) is a flexible mixture model if one takes H as another mixture model.

Huber (1964) initially employed (1.4) for the robust estimation of a location parameter, Zhu and Fukushima (2009) considered generally mixture models concerning the worst-case ES of robust portfolio management. In the spirit of (1.4), Ghosh (2017); Vandewalle et al. (2007) established robust estimations of extreme value index. We remark that model (1.4) is different from the model mis-specification studied by Blanchet and Murthy (2016); Engelke and Ivanovs (2017); Escobar-Bach et al. (2017) concerning the worst VaR and extreme dependence.

The aim of this paper is to study how the common risk measures vary as a function of the properties of the neighbourhood. Basically, we investigate the sensitivity of VaR, ES and the expectile-quantile transformation level w_α, the so-called tail-related risk measures in our context since they are frequently employed in practice with small α level, i.e., modelling the tail of the loss, if the underlying distribution function F_ϵ deviates from the pre-supposed ideal model F in the framework of Huber (1964), that is, F_ϵ is in the ϵ-neighborhood of F specified in (1.4). Our methodology is from extreme value theory, a powerful tool in financial risk management. Therefore, a common assumption is that both F and H belong to the max-domain attraction, i.e., the linear normalization of sample maxima possesses a non-degenerate limit distribution. Further, we suppose that H has a heavier left tail than F, i.e.,

$$\lim_{x \to -\infty} \frac{F(x)}{H(x)} = 0, \tag{1.5}$$

We refer to McNeil et al. (2015) for the monograph of heavy tail analysis in finance and insurance fields.

The contributions of this paper are as follows: a) Sensitivities of common risk measures including VaR, ES and expectile-quantile transformation level are systematically studied and compared; b) Effects on VaR, ES and expectile-quantile transformation level of an infinitesimal contamination to a known F, are investigated by use of influence functions; c) Efficiency of the theoretical results is illustrated by several typical examples and numerical study; d) As an application of the sensitivity of VaR and ES, empirical study involved in the CRIX index, a benchmark of the cyptocurrency market by Trimborn and Härdle (2016); Chen et al. (2017), is given. We expect our research would be beneficial to both financial practitioners and theoretical experts focusing on risk management and extreme value statistics.

The paper is outlined as follows. Section 2 is devoted to establishing the sensitivity of tail-related risk measures in the framework of robustness analysis of Huber (1964). Several illustrating examples are given in Section 3 followed by a small-scale numerical study and an empirical study concerning the CRIX. Proofs are postponed to Section 6.

2 Main Results

Throughout the paper, we keep q_α for the VaR of F and write $q_\alpha(\epsilon)$ for VaR of the contamination model F_ϵ. The same understanding is for $e_\alpha, ES_\alpha, w_\alpha$ and LPM_α involved in (1.2) and (1.3). Further, we write

$$h_1 \sim h_2 \quad \text{or} \quad h_1 = h_2 \{1 + o(1)\}$$

if two functions $h_i(\cdot), i = 1, 2$ such that h_1/h_2 goes to 1 as the argument takes limits.

Our first result below concerns extreme approximations of VaR, ES and w_α when the underlying cdf is contaminated by a heavier tail loss H with a fixed level $\epsilon \in (0, 1]$, see (1.4).

Theorem 2.1 *Consider the contamination model* (1.4) *with $\epsilon \in (0, 1]$ given. Suppose that F and H satisfying* (1.5) *are continuous with infinite left endpoint and finite means. We have as $\alpha \to 0$*

$$q_\alpha(\epsilon) \sim q_{\alpha'}(1) \quad \textit{with} \quad \alpha' \stackrel{\text{def}}{=} \alpha/\epsilon.$$

Further, if the α quantile equals the $w_\alpha(\epsilon)$ expectile, then

$$ES_\alpha(\epsilon) \sim \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}H(x)}{\alpha'}, \quad w_\alpha(\epsilon) \sim \epsilon \left\{ \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha' \right\}.$$

Remark 2.2 *a) We see that once the reference model F is contaminated by a heavier tail distribution H, the tail event involved will be completely determined by the contamination risk H with a scaled probability level α/ϵ. This should be taken as a caveat for the practitioners when they believe the underlying cdf might tend to a known cdf F.*
b) If

$$\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}H(x) \sim \int_{-\infty}^{q_{\alpha'}(1)} x \,\mathrm{d}H(x) \stackrel{\text{def}}{=} LPM_{\alpha'}(1), \tag{2.1}$$

then

$$ES_\alpha(\epsilon) \sim ES_{\alpha'}(1), \quad w_\alpha(\epsilon) \sim \epsilon w_{\alpha'}(1)$$

implying that the ratio w_α/α satisfies that

$$\frac{w_\alpha(\epsilon)}{\alpha} \sim \frac{w_{\alpha'}(1)}{\alpha'}.$$

c) Since $LPM_\alpha \to 0$ as $\alpha \to 0$, the ratio w_α/α satisfies that

$$\frac{w_\alpha}{\alpha} \sim \frac{LPM_\alpha - q_\alpha \alpha}{q_\alpha \alpha} = \mathsf{E}\left[(X/q_\alpha - 1)|\{X < q_\alpha\}\right]$$

depicting the relative distance of the underlying loss X from the α quantile at the left tail. Therefore, the heavier the underlying loss is, the bigger the ratio w_α/α becomes for sufficiently small α.

Note that (2.1) holds for instance when H exhibits considerably heavier tail than typically selected distributions in practice such that $H(x) = |x|^{-\tau}\ell(x)$, $\tau > 1$ with $\ell(\cdot)$ a slowly varying function, that is, $\ell(tx) \sim \ell(t)$, $x > 0$ as $t \to -\infty$, see Example 3.3 below.

According to the latest revisions of the Basel Accords, the risk level α should be determined by the risk measure without changing too much its resulting value and the corresponding capital requirements, Bellini and Di Bernardino (2017). A natural question arising is how the tail-related risk measures vary with infrequent catastrophe losses. Therefore, we explore below the sensitiveness of tail-related risk measures at small risk level α by considering the ϵ-neighborhood

$$\mathcal{F}_\epsilon = \{F_\epsilon | F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)\}$$

with $\epsilon = \epsilon_\alpha \to 0$ as $\alpha \to 0$. We keep the same notation aforementioned with ϵ varying in α.

Theorem 2.3 *Under the same assumptions of Theorem 2.1, suppose further that $\epsilon = \epsilon_\alpha \to 0$ as $\alpha \to 0$.*

a). If H is much heavier than F and/or $\epsilon = \epsilon_\alpha \to 0$ very slowly, such that $F\{q_\alpha(\epsilon)\} = o(1)\epsilon H\{q_\alpha(\epsilon)\}$, i.e., $F\{q_\alpha(\epsilon)\}/[\epsilon H\{q_\alpha(\epsilon)\}] \to 0$ as $\alpha \to 0$, then we have $\lim_{\alpha\to 0} \alpha/\epsilon = 0$, and with $\alpha' \overset{\text{def}}{=} \alpha/\epsilon$

$$q_\alpha(\epsilon) \sim q_{\alpha/\epsilon}(1), \quad ES_\alpha(\epsilon) \sim \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}H(x)}{\alpha'}, \quad w_\alpha(\epsilon) \sim \epsilon\left\{\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha'\right\}.$$

b). If H is to some extend heavier than F and/or $\epsilon = \epsilon_\alpha \to 0$ with certain convergence, such that

$F\{q_\alpha(\epsilon)\} \sim c'\epsilon H\{q_\alpha(\epsilon)\}$ *with some* $c' > 0$*, then*

$$
\begin{aligned}
q_\alpha(\epsilon) &\sim q_{\alpha/\{(c'+1)\epsilon\}}(1) \sim q_{\alpha c'/(c'+1)} \\
ES_\alpha(\epsilon) &\sim \frac{(1+1/c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{\alpha} \sim \frac{(1+c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{\alpha'} \\
w_\alpha(\epsilon) &\sim \frac{(1+1/c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{q_\alpha(\epsilon)} - \alpha \sim \frac{(1+c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha.
\end{aligned}
$$

c). If H *is slightly heavier than* F *and/or* $\epsilon = \epsilon_\alpha \to 0$ *very quickly such that* $\epsilon H\{q_\alpha(\epsilon)\} = o(1)F\{q_\alpha(\epsilon)\}$*, then*

$$
q_\alpha(\epsilon) \sim q_\alpha, \quad ES_\alpha(\epsilon) \sim \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{\alpha}, \quad w_\alpha(\epsilon) \sim \left\{\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{q_\alpha(\epsilon)} - \alpha\right\}.
$$

Remark 2.4 *a) Theorem 2.3 covers all three cases by changing the roles between* $F(x)$ *and* $\epsilon H(x)$ *with* $x = q_\alpha(\epsilon) \to -\infty$ *as* $\alpha \to 0$.

b) Case a) indicates the same claim as for ϵ *given, and case c) implies that the tail-related risk measures are robust with very slight contamination level* ϵ_α *for instance* $\epsilon_\alpha = c\alpha^\tau$ *for some* $c > 0$ *and all* $\tau \geq 1$.

c) A typical example of contamination model $(1-\epsilon)\Phi(x\sqrt{1-\epsilon}) + \epsilon\Phi((x-\mu)\sqrt{\epsilon})$ *with* μ *a constant, and* $\Phi(\cdot)$ *the standard normal cdf, discussed in Kuan et al. (2009), gives different sensitivity with respect to the contamination level* $\epsilon = \epsilon_\alpha$.

The influence function approach, known also as the "infinitesimal approach", is generally employed to give qualitative robustness measure, for instance Fermanian and Scaillet (2005) investigates robust risk portfolios under netting agreements when the level of contamination in the data gradually decreases to zero. Recall that the influence function of some risk measure ϱ is defined as follows.

$$
IF(\varrho; F, H) = \lim_{\epsilon \to 0} \frac{\varrho(\epsilon) - \varrho(0)}{\epsilon} = \frac{\partial \varrho(\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0}
$$

with $\varrho(\epsilon)$ standing for the risk measure ϱ of the contamination model $F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)$.

Below, we study the influence function (IF) of VaR and ES evaluating its approximate bias if the corresponding risk measures are based loosely on the pre-supposed idea model F.

Theorem 2.5 *Assume that F has positive continuous differential at the α quantile, and H is continuous at q_α. We have*

$$IF(q_\alpha; F, H) = \frac{\alpha - H(q_\alpha)}{F'(q_\alpha)}, \qquad IF(ES_\alpha; F, H) = \frac{q_\alpha\{\alpha - H(q_\alpha)\} + \int_{-\infty}^{q_\alpha} x \,\mathrm{d}\{H(x) - F(x)\}}{\alpha}.$$

Remark 2.6 *a) Note that if H has heavier left tail than F, i.e., (1.5) holds, then in view of Theorem 2.5, both influence functions are negative for small α. Further, it holds for sufficiently small ϵ,*

$$q_\alpha \simeq q_\alpha(\epsilon) - \epsilon IF(q_\alpha; F, H), \qquad ES_\alpha \simeq ES_\alpha(\epsilon) - \epsilon IF(ES_\alpha; F, H).$$

We conclude that both VaR and ES based on the referenced model F have a slightly positive bias in comparison to those strictly on the contamination model.

b) The influence function leads to some robustness measure such as gross error sensitivity of the estimation of the tail-related risk measure through the following worst-case scenario

$$\sup_{H \in \mathcal{H}} |IF(\varrho; F, H)| \quad or \quad \sup_{H \in \mathcal{H}} |IF(\varrho; F, H)|^2,$$

where $\mathcal{H}$ is the class of contamination models H's. This might evoke the min-max global robustness analysis in risk management, Brazauskas (2003).

3 Examples

We illustrate our theoretical results through three examples with the standard normal cdf being the pre-supposed ideal model F, and H being the normal, Laplace and Power-like distributions in Examples 3.1-3.3, subsequently.

Example 3.1 *Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon\Phi(x/\sigma)$ with $\sigma > 1$, the scale parameter of the contamination model $H(x) = \Phi(x/\sigma)$(the same as below). Clearly, condition (1.5) holds, and the larger σ is,* *the heavier tail H possesses. Recall $q_\alpha(\epsilon)$ denotes the α quantile of F_ϵ. We have, with $\alpha' = \alpha/\epsilon \to 0$*

$$q_\alpha(\epsilon) \sim q_{\alpha'}(1) = \sigma\Phi^{-1}(\alpha'), \quad \int_{-\infty}^{u} x\, \mathrm{d}\Phi(x/\sigma) = -\sigma\varphi(u/\sigma), \quad u \to -\infty$$

with $\varphi(\cdot)$ standing for the probability density function (pdf) of a standard normal random variable. Further,

$$ES_\alpha(\epsilon) \;=\; -\frac{(1-\epsilon)\varphi\{q_\alpha(\epsilon)\} + \epsilon\sigma\varphi\left\{q_\alpha(\epsilon)/\sigma\right\}}{\alpha} \sim -\frac{\sigma\varphi\left\{q_\alpha(\epsilon)/\sigma\right\}}{\alpha'}.$$

Further, a straightforward calculation yields by setting $u = q_\alpha(\epsilon)$:

$$\begin{aligned} w_\alpha(\epsilon) &= \frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma) + u\alpha}{2\left\{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma) + u\alpha\right\} - u} \\ &\sim -\frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{u} - \alpha \\ &\sim \epsilon\left\{-\frac{\sigma\varphi(u/\sigma)}{u} - \alpha'\right\}. \end{aligned}$$

Next, we consider the case $\epsilon = \epsilon_\alpha$. It follows by the Mills' ratio $\Phi(x) \sim \varphi(x)/|x|$, $x \to -\infty$ that, for $\epsilon = \sqrt{\alpha}$ and $1 < \sigma^2 \le 2$

$$ES_\alpha(\epsilon) = -\frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{\alpha} \sim -\frac{\varphi(u)}{\alpha},$$

where $u = q_\alpha(\epsilon) \sim \Phi^{-1}(\alpha)$.

Similarly, we have for $\epsilon = \sqrt{\alpha}$ and $\sigma^2 > 2$

$$ES_\alpha(\epsilon) = -\frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{\alpha} \sim -\frac{\sigma\varphi(u/\sigma)}{\alpha'},$$

where $u = q_\alpha(\epsilon) \sim \sigma\Phi^{-1}(\alpha/\epsilon)$.

We conclude the sensitivity of VaR, ES and w_α via the contamination level ϵ and the heaviness parameter σ, coinciding the claims established in Theorems 2.1 and 2.3.

Example 3.2 *Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon L(\sqrt{2}x/\sigma)$, $\sigma > 0$ with $L(\cdot)$ the standard Laplace distri-*

bution (double-sided exponential distribution), i.e., the density function $l(\cdot)$ is given by

$$l(x) = \frac{1}{2} \exp\{-|x|\}, \quad x \in \mathbb{R}.$$

It follows that (1.5) *holds, and for $\alpha < 0.5, u < 0$*

$$L^{-1}(\alpha) = \log(2\alpha), \quad \int_{-\infty}^{u} x \, \mathrm{d}L(x) = \frac{1}{2}(u-1)\mathrm{e}^{u} \stackrel{\text{def}}{=} LP(u).$$

Clearly, with $\alpha' = \alpha/\epsilon$

$$q_\alpha(\epsilon) \sim q_{\alpha'}(1) = \frac{\sigma}{\sqrt{2}} L^{-1}(\alpha') = \frac{\sigma}{\sqrt{2}} \log(2\alpha'), \quad \alpha \to 0.$$

and

$$\begin{aligned} ES_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi\{q_\alpha(\epsilon)\} + \epsilon\sigma/\sqrt{2}LP\left\{\sqrt{2}q_\alpha(\epsilon)/\sigma\right\}}{\alpha} \\ &\sim \frac{\sigma}{2\sqrt{2}\alpha'} \left\{\frac{\sqrt{2}}{\sigma} q_\alpha(\epsilon) - 1\right\} \exp\left\{\frac{\sqrt{2}}{\sigma} q_\alpha(\epsilon)\right\}. \end{aligned}$$

Further, we have by setting $u = q_\alpha(\epsilon)$:

$$\begin{aligned} w_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma) - u\alpha}{2\left\{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma) - u\alpha\right\} + u} \\ &\sim \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma)}{u} - \alpha \\ &\sim \epsilon\left\{\frac{LP(\sqrt{2}u/\sigma)}{\sqrt{2}u/\sigma} - \alpha'\right\}. \end{aligned}$$

Next, we consider the case $\epsilon = \epsilon_\alpha$. It follows by elementary calculations that, for $\epsilon = \sqrt{\alpha}$,

$$\begin{aligned} ES_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP\left(\sqrt{2}u/\sigma\right)}{\alpha} \\ &\sim \frac{\sigma}{2\sqrt{2}\alpha'} \left(\frac{\sqrt{2}}{\sigma} u - 1\right) \exp\left\{\frac{\sqrt{2}}{\sigma} u\right\}, \end{aligned}$$

where $u = q_\alpha(\epsilon) \sim q_{\alpha'}(1) = (\sigma/\sqrt{2}) \log(2\sqrt{\alpha})$.

If we take $\epsilon = \alpha$, we have $u = q_\alpha(\epsilon) \sim q_\alpha = \Phi^{-1}(\alpha)$ and

$$ES_\alpha(\epsilon) = \frac{-(1-\epsilon)\varphi(u) + \epsilon(\sigma/\sqrt{2})LP\left(\sqrt{2}u/\sigma\right)}{\alpha} \sim -\frac{\varphi(u)}{\alpha}.$$

Comparing the Laplace with normal contamination models, we see that the risk measures are more sensitive for heavier contamination models with even infrequent contamination data. Financial practitioners should therefore take care of the extreme value risk due to asset-specific events and market-wide events.

Example 3.3 *Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon H(x/\sigma)$, $\sigma > 0$, with H a symmetric distribution such that*

$$H(x) = \frac{1}{2}\left\{1 - \left(1 - \frac{4}{4+x^2}\right)^{0.5}\right\}, \quad x < 0.$$

Clearly, Remark 2.2 a) holds with $H(x) \sim |x|^{-2}/2$ as $x \to -\infty$, i.e., H decays slowly like a power function with index -2. Hence, H is the so-called Power-like distribution with scale parameter $\sigma > 0$ in the context. Further, with $\alpha' = \alpha/\epsilon$

$$q_\alpha(\epsilon) \sim q_{\alpha'}(1) = -\frac{1-2\alpha'}{\sqrt{\alpha'(1-\alpha')}}, \quad \alpha \to 0 \tag{3.1}$$

$$\int_{-\infty}^{u} x \, \mathrm{d}H(x) = \int_{-\infty}^{u} \frac{2x}{(4+x^2)^{3/2}} \, \mathrm{d}x = -\frac{2}{\sqrt{4+u^2}} \stackrel{\text{def}}{=} LP(u).$$

We see that $|LP(\cdot)|$ is a regular varying function at $-\infty$ with index -1. Therefore,

$$\begin{aligned} ES_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi\{q_\alpha(\epsilon)\} + \epsilon\sigma LP\{q_\alpha(\epsilon)/\sigma\}}{\alpha} \\ &\sim \frac{\sigma LP\{q_\alpha(\epsilon)/\sigma\}}{\alpha'} \sim \frac{\sigma LP\{q_{\alpha'}(1)/\sigma\}}{\alpha'} = ES_{\alpha'}(1). \end{aligned}$$

Further, we have by setting $u = q_\alpha(\epsilon)$

$$\begin{aligned} w_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma) - u\alpha}{2\left\{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma) - u\alpha\right\} + u} \\ &\sim \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma)}{u} - \alpha \end{aligned}$$

$$\sim \epsilon \left\{ \frac{\sigma LP(u/\sigma)}{u} - \alpha' \right\} \sim \epsilon w_{\alpha'}(1).$$

Next, we consider two cases of $\epsilon = \epsilon_\alpha$ so that the tail-related risk measures determined by F and H, respectively.

- *For $\epsilon = \sqrt{\alpha}$ tending to zero slowly, we have with $u = q_\alpha(\epsilon) \sim q_{\sqrt{\alpha}}(1)$ given by (3.1)*

$$ES_\alpha(\epsilon) = \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma)}{\alpha} \sim \frac{\sigma LP(u/\sigma)}{\sqrt{\alpha}} \sim ES_{\alpha'}(1).$$

- *For $\epsilon = \alpha$, we have*

$$ES_\alpha(\epsilon) \sim -\frac{\varphi(u)}{\alpha}, \quad u = q_\alpha(\epsilon) \sim q_\alpha = \Phi^{-1}(\alpha).$$

We remark that the Power-like contamination distribution is a typical example that the probability level ratio $w_\alpha/\alpha = 1$. For a realistic degree of tail heaviness, the ratio w_α/α is less than 1, and increases with the degree of tail heaviness, Mihoci et al. (2017).

4 Numerical study

In this section, we investigate the behavior of the theoretical discussions by Theorems 2.1, 2.3 and 2.5 with the three examples given by Section 3.

In Fig. 1, we fix the contamination level ϵ to be 0.5. In view of Theorem 2.1, the approximations of the VaR, ES and the expectile-quantile transformation level ratio are given by the heavier contamination distribution at level $\alpha' = \alpha/\epsilon$, i.e., $q_{\alpha'}(1), ES_{\alpha'}(1)$ and $w_{\alpha'}(1)/\alpha'$. As the risk level $\alpha \to 0$, the more accurate approximations to the true values based strictly on the underlying model are obtained.

In Fig. 2, we fix small risk level $\alpha = 0.5\%$ and investigate how the approximations vary with the contamination level ϵ. Clearly, the VaR, ES and the probability level ratio w_α/α becomes smaller and smaller as ϵ is closer to 1, in other words, the smaller ratio level indicates the heavier left tails of the contamination model. Further, the approximations performs better for larger ϵ. Finally, we conclude that the level ratio of expectile vs. quantile is not monotonic for moderate ϵ, and the normal-Power-like contamination model have ratio level around 1.

In Fig. 3, we conduct the numerical approximations based on the reference normal model with small contamination level $\epsilon = \alpha^\tau$ and small scale parameter σ of the contamination model H. Generally, the efficiency of approximations supports the claim in c) of Theorem 2.3, and the slower rate of approximations of ES than that of VaR since ES catches the tails of the loss, and therefore the slowest rate of approximations is given with the heaviest Power-like model.

In comparison to Fig. 1 and Fig. 3, we investigate, in Fig 4, the approximations based on the heavier contamination model H and suitable contamination level $\epsilon_\alpha \to 0$ as $\alpha \to 0$. Conversely, we get that the approximations perform better with heavier contamination models, and the efficiency of approximations for ES is more obvious than that for VaR.

Finally, in view of Theorem 2.5, we estimate the risk measure ϱ by $\widetilde{\varrho}(\epsilon) \overset{\text{def}}{=} \varrho + \epsilon IF(\varrho; F, H)$ for small ϵ, provided that both F and H are asymptotically known. Define the relative error (RE) of the estimations as follows.

$$\mathrm{RE}(\varrho) = \frac{\widetilde{\varrho}(\epsilon) - \varrho(\epsilon)}{\varrho(\epsilon)}. \tag{4.1}$$

We see from Table 1 that, the smaller ϵ is, the less RE is. Further, the RE of ES is in general larger than that of VaR since ES gives the essential tail expectation which is more influenced by the heavy-tailed contamination model. Therefore, we conclude that Theorem 2.5 gives nice estimations of VaR as well as ES for general α.

Normal										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
q_α	67.49	67.86	68.24	68.62	69.01	69.40	69.79	70.18	70.59	71.00
ES_α	127.25	128.68	130.10	131.52	132.94	134.36	135.77	137.19	138.60	140.01
RE(q_α)	−0.02	−0.06	−0.16	−0.31	−0.53	−0.80	−1.13	−143	−187	−2.37
RE(ES_α)	0.01	0.02	0.04	0.08	0.13	0.19	0.27	0.32	0.42	0.53
Laplace										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
q_α	67.44	67.37	67.29	67.21	67.14	67.06	66.98	66.90	66.83	66.75
ES_α	127.13	127.30	127.47	127.64	127.81	127.97	128.14	128.31	128.48	128.65
RE(q_α)	−0.02	−0.02	−0.01	−0.01	0.01	0.02	0.04	0.06	0.08	0.11
RE(ES_α)	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03
Power-like										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
q_α	67.48	67.76	68.05	68.34	68.64	68.93	69.23	69.53	69.83	70.13
ES_α	127.34	129.61	131.89	134.16	136.43	138.70	140.97	143.24	145.51	147.78
RE(q_α)	−0.02	−0.05	−0.11	−0.20	−0.33	−0.50	−0.71	−0.94	−1.22	−1.44
RE(ES_α)	0.01	0.02	0.03	0.05	0.08	0.1.13	0.15	0.20	0.26	0.28

Table 1: Comparisons of the relative errors (RE) of VaR and ES at level $\alpha = 0.25$ with varying level and contamination model H being normal, Laplace and Power-like with mean zero and scale parameter $\sigma = 2, 1.2, 1$. For convenience, the measures are reported as positive numbers, and the unit of $\epsilon, q_\alpha, ES_\alpha$ and $\mathrm{RE}(q_\alpha), \mathrm{RE}(ES_\alpha)$ are % and ‰, respectively.

5 Empirical study on CRIX

The CRIX, a market index (benchmark), is designed by Trimborn and Härdle (2016). It enables each interested party to study the performance of the crypto market as a whole or single cryptos, and therefore attracts increasing attention of risk managers and regulators. Consequently, we focus on its tail feature and give the estimations of the tail-related risk measures. As shown below, this is achieved by using the normal-Laplace contamination model and the approximations given in Section 2. Here, we use the daily CRIX dataset during 2014-07-31-2018-01-01, which is available on http://crix.hu-berlin.de.

Firstly, we explore the distributional feature of the financial dataset CRIX. Clearly, the normal Q-Q plot in Fig. 5 shows that the log returns of CRIX deviate from normal distribution, both tails appear to be heavier than the normal distribution.

Next, in Fig. 6 we employ the empirical mean excess function from extreme value theory to analyse its heaviness:

$$\widehat{m}_X(t) = \frac{\sum_{i=1}^n (X_i - t)\, \mathbf{I}\{X_i > t\}}{\sum_{i=1}^n \mathbf{I}\{X_i > t\}}, \quad t \text{ large},$$

where X_i's is a sample of size n, the observations of X, the daily log returns of CRIX. We see that both of upper and lower tails of CRIX decay exponentially since the graph $(t, \widehat{m}_X(t))$ becomes linear with slope zero for large threshold t. Further, a close look at the horizontal change of the log mean excess graph $(\log t, \log \widehat{m}_X(t))$ for large t, indicates the Laplace tail-decaying of the dataset, Dierckx et al. (2009). We remark that the left tail feature of X is given by the right tail of $-X$. Consequently, we hereafter use the normal-Laplace contamination model with parameter $\epsilon, \boldsymbol{\mu} = (\mu_1, \mu_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ to fit the log returns of CRIX:

$$F_\epsilon(x) = (1-\epsilon) * \Phi\left(\frac{x-\mu_1}{\sigma_1}\right) + \epsilon * L\left(\sqrt{2}\frac{x-\mu_2}{\sigma_2}\right), \quad x \in \mathbb{R}. \tag{5.1}$$

The estimated parameters involved in (5.1) are given in Table 2 by utilizing the expectation-maximization (EM) algorithm, Dempster et al. (1977). Further, for the CRIX during 2016.04.01–2018.01.01 the contamination level $\widehat{\epsilon} = 0.730$ is slightly bigger than the other two periods. We conclude the tail heaviness of CRIX time series might probably have an increasing tendency.

Finally, estimations of VaR and ES at level $\alpha = 0.5\%, 1\%, 5\%$ are compared also in Table 2 via three methods including the historical simulation, written by $\widehat{q}_\alpha^*, \widehat{ES}_\alpha^*$; Laplace approximations at level $\alpha' = \alpha/\widehat{\epsilon}$ by use of Theorem 2.1, denoted by $\widehat{q}_{\alpha'}(1), \widehat{ES}_{\alpha'}(1)$; and approximations based directly on the normal-Laplace mixture model, written by $\widehat{q}_\alpha(\widehat{\epsilon}), \widehat{ES}_\alpha(\widehat{\epsilon})$. For all the estimations of ES, we keep the historical simulations of VaR, as in Mihoci et al. (2017).

We conclude that all the estimations for the sub-period of 2016.04.01–2018.01.01 are slightly larger (in absolute value) than those for the other periods, which might be caused by the heavier tail of the CRIX in that period. Further, the Laplace approximations are rather close to the complete contamination model, illustrating again the efficiency of our theoretical approximation in Theorem 2.1.

period	parameter	α	$\widehat{q}_\alpha^*$	$\widehat{q}_{\alpha'}(1)$	$\widehat{q}_\alpha(\epsilon)$	$\widehat{ES}_\alpha^*$	$\widehat{ES}_{\alpha'}(1)$	$\widehat{ES}_\alpha(\epsilon)$
	$\widehat{\epsilon} = 0.686$	0.5%	0.137	0.128	0.128	0.179	0.130	0.000
2014.07-2018.01	$\widehat{\boldsymbol{\mu}} = (0.001, 0.004)$	1%	0.105	0.103	0.103	0.152	0.129	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.009, 0.043)$	5%	0.054	0.055	0.054	0.091	0.085	0.000
	$\widehat{\epsilon} = 0.564$	0.5	0.137	0.128	0.128	0.179	0.130	0.000
2014.07-2016.04	$\widehat{\boldsymbol{\mu}} = (0.001, -0.002)$	1%	0.104	0.101	0.101	0.143	0.119	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.013, 0.042)$	5%	0.046	0.053	0.053	0.086	0.095	0.000
	$\widehat{\epsilon} = 0.730$	0.5%	0.137	0.128	0.128	0.179	0.130	0.000
2016.04-2018.01	$\widehat{\boldsymbol{\mu}} = (0.002, 0.008)$	1%	0.108	0.106	0.106	0.155	0.130	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.006, 0.045)$	5%	0.059	0.055	0.055	0.094	0.080	0.000

Table 2: Estimated parameters of the normal-Laplace contamination model for the log return CRIX during 2014-7-31–2018-01-01 and the two sub-periods. Here $\widehat{q}_\alpha^*$ and $\widehat{ES}_\alpha^*$ stand respectively for historical simulations of VaR and ES, and $\widehat{q}_{\alpha'}(1), \widehat{q}_\alpha(\epsilon)$ and $\widehat{ES}_{\alpha'}(1), \widehat{ES}_\alpha(\epsilon)$ are those estimations based on Laplace and the original mixture model with estimated parameters involved and $\alpha' = \alpha/\epsilon$.

6 Proofs

Note by (1.2) and (1.3) that $ES_\alpha = LPM_\alpha/\alpha$. Hence, in view of (1.3), it suffices to find $q_\alpha(\epsilon)$ and $LPM_\alpha(\epsilon)$. Since

$$(1-\epsilon)F\left\{q_\alpha(\epsilon)\right\} + \epsilon H\left\{q_\alpha(\epsilon)\right\} = \alpha, \tag{6.1}$$

we have by (1.5)

$$H\left\{q_\alpha(\epsilon)\right\} \sim \alpha/\epsilon.$$

Therefore, it follows further from Lemma 1.2.9 in de Haan and Ferreira (2006) that, for given

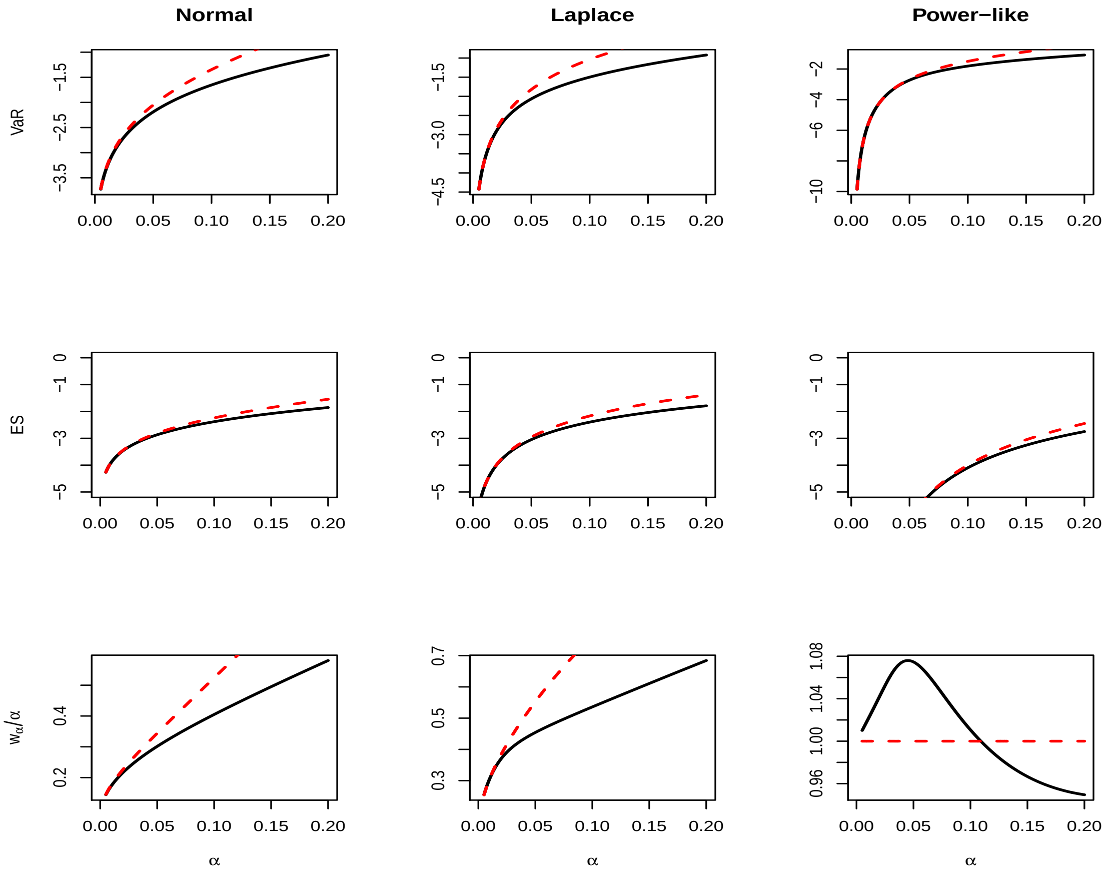

Figure 1: Comparisons of the **true values** (black line) based strictly on the contamination model F_ϵ and **approximations** (red dotted line) based loosely on the contamination distribution H at level $\alpha' = \alpha/\epsilon$ via Value-at-Risk, Expected-shortfall and the expectile-quantile transformation level ratio w_α/α. Here $\epsilon = 0.5$ and H is the normal, Laplace and Power-like distribution with scale parameter $\sigma = 1.6, 1.6, 1$, accordingly.

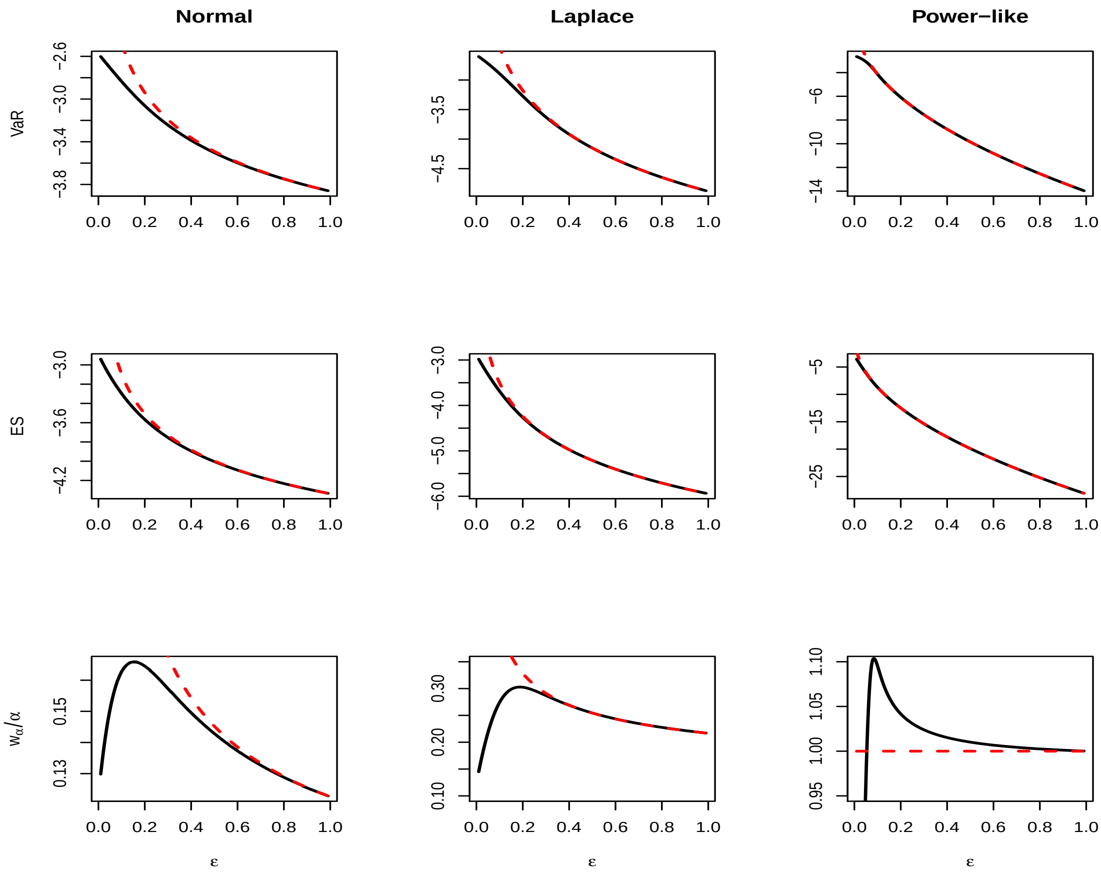

Figure 2: Comparisons of the **true values** (black line) and **approximations** (dotted line) based on the contamination model H at level α/ϵ via Value-at-Risk, Expected-shortfall and the expectile-quantile transformation level ratio w_α/α. Here $\alpha = 0.5\%$, and ϵ varies in $(0.01, 1)$, and H is normal, Laplace and Power-like distribution with the scale parameter $\sigma = 1.5, 1.5, 1$, accordingly.

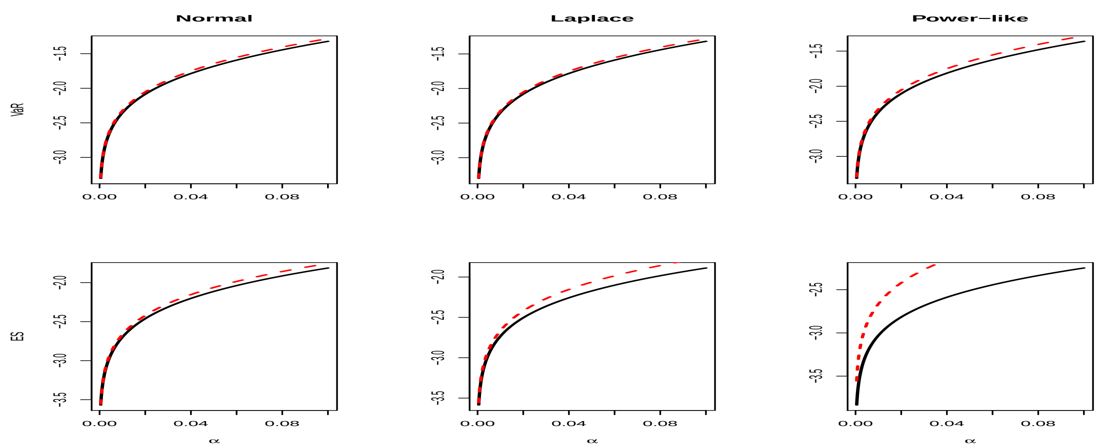

Figure 3: Comparisons of the **true values** (black line) and **approximations** (dotted line) based on the pre-supposed ideal model $F(x) = \Phi(x)$. Here H is normal and Laplace, Power-like distributions with scale parameter $\sigma = 1.1, 1.6, 1.0$ and $\epsilon = \alpha^\tau$ with $\tau = 0.5, 1, 1$, accordingly.

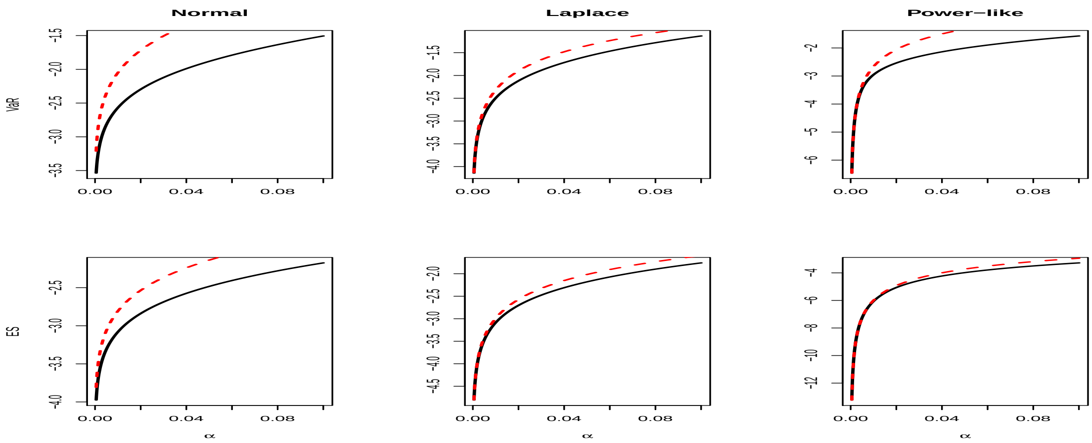

Figure 4: Comparisons of the **true values** (black line) and **approximations** (dotted line) based on the heavier tail contamination model H. Here H is normal and Laplace, Power-like distributions with scale parameter $\sigma = 1.6, 0.95, 1$ and contamination level $\epsilon = \alpha^{\tau}$ with $\tau = 0.5, 0.1, 0.5$, accordingly.

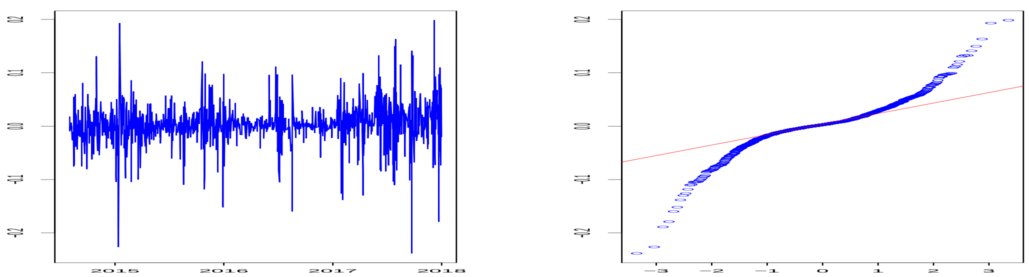

Figure 5: Time series of daily log returns of CRIX during 2014.07.31 – 2018.01.01 (left) and normal Q-Q plot (right).

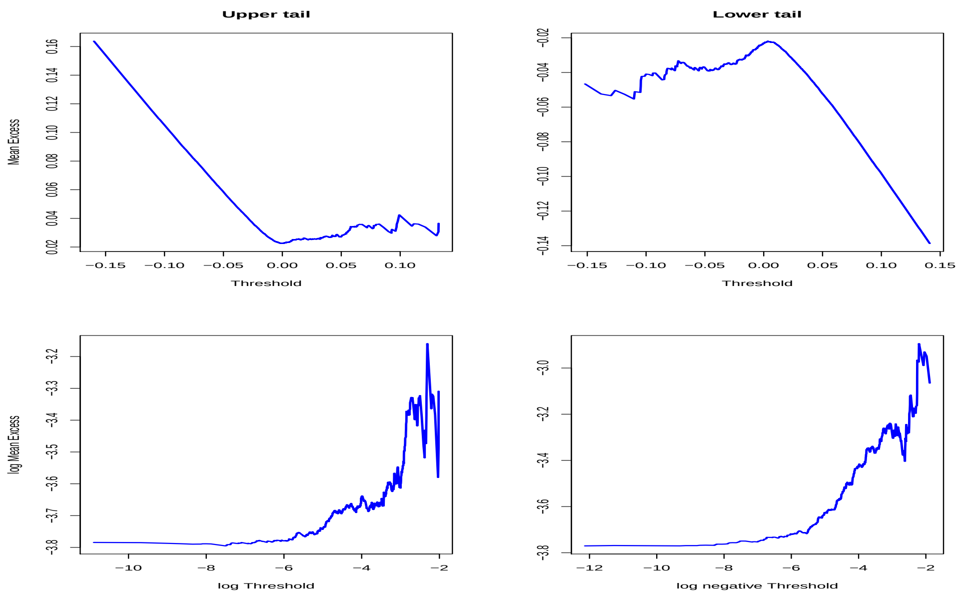

Figure 6: Mean Excess plot and log Mean Excess plot for log returns of CRIX during 2014.07.31 – 2018.01.01.

$\epsilon \in (0, 1]$,

$$q_\alpha(\epsilon) \sim q_{\alpha/\epsilon}(1), \tag{6.2}$$

that is, the VaR for the contamination model equals approximately the VaR of H (the heavier df) at probability level α/ϵ.

Next, we return to $LPM_\alpha(\epsilon)$. Note that by a straightforward application of L'Hôpital' rule into (1.5)

$$\begin{aligned} LPM_\alpha(\epsilon) &= (1-\epsilon)\int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}F(x) + \epsilon \int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}H(x) \\ &\sim \epsilon \int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}H(x), \end{aligned} \tag{6.3}$$

which goes to 0 by the fact that $\mathsf{E}[Y] < \infty$ and $q_\alpha(\epsilon) \to -\infty$.

Therefore, in view of (1.2), we have (recall $\alpha' = \alpha/\epsilon$)

$$w_\alpha(\epsilon) \sim \frac{LPM_\alpha(\epsilon)}{q_\alpha(\epsilon)} - \alpha \sim \epsilon \left\{ \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}H(x)}{q_{\alpha'}(1)} - \alpha' \right\} \tag{6.4}$$

Consequently, the claim follows by (6.2)-(6.4).

Proof of Theorem 2.3 Clearly, we have $\lim_{\alpha\to 0} F\{q_\alpha(\epsilon)\} = 0$ since $\lim_{\alpha\to 0} \max(\alpha, \epsilon_\alpha) = 0$ and $q_\alpha(\epsilon)$ satisfies (6.1). Hence, $\lim_{\alpha\to 0} q_\alpha(\epsilon) = -\infty$.

a) If $F\{q_\alpha(\epsilon)\} = o(1)\epsilon H\{q_\alpha(\epsilon)\}$, then

$$\epsilon H\{q_\alpha(\epsilon)\} \sim (1-\epsilon)F\{q_\alpha(\epsilon)\} + \epsilon H\{q_\alpha(\epsilon)\} = \alpha.$$

Recalling that H is in the max-domain attraction, it follows by Lemma 1.2.9 by de Haan and Ferreira (2006) that

$$q_\alpha(\epsilon) \sim q_{\alpha/\epsilon}(1).$$

b) If $F\{q_\alpha(\epsilon)\} \sim c'\epsilon H\{q_\alpha(\epsilon)\}$ with some fixed $c' > 0$, then

$$q_\alpha(\epsilon) \sim q_{\alpha/\{(c'+1)\epsilon\}}(1) \sim q_{\alpha c'/(c'+1)}.$$

c) If $\epsilon H\{q_\alpha(\epsilon)\} = o(1)F\{q_\alpha(\epsilon)\}$, then

$$F\{q_\alpha(\epsilon)\} \sim (1-\epsilon)F\{q_\alpha(\epsilon)\} + \epsilon H\{q_\alpha(\epsilon)\} = \alpha$$

implying that $q_\alpha(\epsilon) \sim q_\alpha$.

Proof of Theorem 2.5 First, for given $\alpha \in (0,1)$, denote by θ the influence function of VaR with the underlying df in the ϵ-neighbourhood $\mathcal{F}_\epsilon = \{F_\epsilon | F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)\}$. We have

$$q_\alpha(\epsilon) \simeq q_\alpha + \epsilon\theta.$$

Therefore, we have by Taylor's expansion of F at q_α that

$$\lim_{\epsilon\to 0}\left[(1-\epsilon)\{\alpha + \epsilon\theta F'(q_\alpha)\} + \epsilon H(q_\alpha)\right] = \alpha.$$

This implies that $\theta F'(q_\alpha) + H(q_\alpha) = \alpha$. The first claim is obtained.

Similarly, we have for the ES

$$
\begin{aligned}
IF(ES_\alpha; F, H) &= \lim_{\epsilon\to 0} \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \,\mathrm{d}F_\epsilon(x) - \int_{-\infty}^{q_\alpha} x \,\mathrm{d}F(x)}{\alpha\epsilon} \\
&= \frac{\int_{-\infty}^{q_\alpha} x \,\mathrm{d}\{H(x) - F(x)\} + \lim_{\epsilon\to 0} \epsilon^{-1} \int_{q_\alpha}^{q_\alpha(\epsilon)} x \,\mathrm{d}F_\epsilon(x)}{\alpha} \\
&= \frac{\int_{-\infty}^{q_\alpha} x \,\mathrm{d}\{H(x) - F(x)\} + q_\alpha\{\alpha - H(q_\alpha)\}}{\alpha},
\end{aligned}
$$

where the last step follows by $q_\alpha(\epsilon) - q_\alpha \simeq \theta\epsilon$ and the continuity of F' at q_α.

Acknowledgments. The authors would like to thank the referee for his\her important suggestions which significantly improve this contribution. The authors also would like to thank Prof. Enkelejd Hashorva for useful comments when C. Ling worked on this project during visit of University of Lausanne, Switzerland.

References

Artzner, P., Delbaen, F., Eber, J. and Heath, D. (1999). Coherent measures of risk, *Mathematical Finance* **9**(3): 203–228.

Bellini, F. and Di Bernardino, E. (2017). Risk management with expectiles, *The European Journal of Finance* **23**(6): 487–506.

Blanchet, J. and Murthy, K. (2016). On distributionally robust extreme value analysis, *arXiv preprint arXiv:1601.06858* .

Borke, L. and W.K., H. (2018). *Q3-D3-LSA, Handbook of Big data Analytics*, Springer Verlag.

Brazauskas, V. (2003). Influence functions of empirical nonparametric estimators of net reinsurance premiums, *Insurance: Mathematics and Economics* **32**(1): 115–133.

Chen, S., Chen, C., Härdle, W., Lee, T. and Ong, B. (2017). Econometric analysis of a cryptocurrency index for portfolio investment, *Handbook of Blockchain, Digital Finance, and Inclusion, Volume 1*, Elsevier, pp. 175–206.

Cont, R., Deguest, R. and Scandolo, G. (2010). Robustness and sensitivity analysis of risk measurement procedures, *Quantitative finance* **10**(6): 593–606.

Daouia, A., Girard, S. and Stupfler, G. (2017). Estimation of tail risk based on extreme expectiles, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* .

de Haan, L. and Ferreira, A. (2006). *Extreme value theory*, Springer Series in Operations Research and Financial Engineering, Springer, New York.

Delbaen, F. (2002). Coherent risk measures on general probability spaces, *Advances in finance and stochastics*, Springer, pp. 1–37.

Dempster, A., Laird, N. and Rubin, D. (1977). Maximum likelihood from incomplete data via the em algorithm, *Journal of the royal statistical society. Series B (methodological)* pp. 1–38.

Dierckx, G., Beirlant, J., De Waal, D. and Guillou, A. (2009). A new estimation method for Weibull-type tails based on the mean excess function, *Journal of Statistical Planning and Inference* **139**(6): 1905–1920.

Engelke, S. and Ivanovs, J. (2017). Robust bounds in multivariate extremes, *The Annals of Applied Probability* **27**(6): 3706–3734.

Escobar-Bach, M., Goegebeur, Y., Guillou, A. and You, A. (2017). Bias-corrected and robust estimation of the bivariate stable tail dependence function, *TEST* **26**(2): 284–307.

Fermanian, J. and Scaillet, O. (2005). Sensitivity analysis of var and expected shortfall for portfolios under netting agreements, *Journal of Banking & Finance* **29**(4): 927–958.

Ghosh, A. (2017). Divergence based robust estimation of the tail index through an exponential regression model, *Statistical Methods & Applications* **26**(2): 181–213.

Huber, P. (1964). Robust estimation of a location parameter, *The Annals of Mathematical Statistics* **35**(1): 73–101.

Jones, M. (1994). Expectiles and M-quantiles are quantiles, *Statistics & Probability Letters* **20**(2): 149–153.

Kuan, C., Yeh, J. and Hsu, Y. (2009). Assessing value at risk with CARE, the conditional autoregressive expectile models, *Journal of Econometrics* **150**(2): 261–270.

McNeil, A., Frey, R. and Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools*, Princeton university press.

Mihoci, A., Härdle, W. and Chen, C. (2017). TERES-Tail event risk expectile based on shortfall, *submitted to Quantitative Finance* pp. 1–13.

Newey, W. and Powell, J. (1987). Asymmetric least squares estimation and testing, *Econometrica* **55**(4): 819–847.

Nolde, N., Ziegel, J. F. et al. (2017). Elicitability and backtesting: Perspectives for banking regulation, *The Annals of Applied Statistics* **11**(4): 1833–1874.

Taylor, J. (2008). Estimating value at risk and expected shortfall using expectiles, *Journal of Financial Econometrics* **6**(2): 231–252.

Trimborn, S. and Härdle, W. (2016). CRIX an index for blockchain based currencies, *SFB 649 Discussion Paper 2016-021 submitted to Journal of Empirical Finance* .

Vandewalle, B., Beirlant, J., Christmann, A. and Hubert, M. (2007). A robust estimator for the tail index of Pareto-type distributions, *Computational Statistics & Data Analysis* **51**(12): 6252–6268.

Xu, X., Mihoci, A. and Härdle, K. (2018). lcare-localizing conditional autoregressive expectiles, *Journal of Empirical Economics* **to appear**.

Yao, Q. and Tong, H. (1996). Asymmetric least squares regression estimation: A nonparametric approach, *Journal of Nonparametric Statistics* **6**(2-3): 273–292.

Zhu, S. and Fukushima, M. (2009). Worst-case conditional value-at-risk with application to robust portfolio management, *Operations research* **57**(5): 1155–1168.

IRTG 1792 Discussion Paper Series 2018

For a complete list of Discussion Papers published, please visit irtg1792.hu-berlin.de.

001 "Data Driven Value-at-Risk Forecasting using a SVR-GARCH-KDE Hybrid" by Marius Lux, Wolfgang Karl Härdle and Stefan Lessmann, January 2018.

002 "Nonparametric Variable Selection and Its Application to Additive Models" by Zheng-Hui Feng, Lu Lin, Ruo-Qing Zhu asnd Li-Xing Zhu, January 2018.

003 "Systemic Risk in Global Volatility Spillover Networks: Evidence from Option-implied Volatility Indices " by Zihui Yang and Yinggang Zhou, January 2018.

004 "Pricing Cryptocurrency options: the case of CRIX and Bitcoin" by Cathy YH Chen, Wolfgang Karl Härdle, Ai Jun Hou and Weining Wang, January 2018.

005 "Testing for bubbles in cryptocurrencies with time-varying volatility" by Christian M. Hafner, January 2018.

006 "A Note on Cryptocurrencies and Currency Competition" by Anna Almosova, January 2018.

007 "Knowing me, knowing you: inventor mobility and the formation of technology-oriented alliances" by Stefan Wagner and Martin C. Goossen, February 2018.

008 "A Monetary Model of Blockchain" by Anna Almosova, February 2018.

009 "Deregulated day-ahead electricity markets in Southeast Europe: Price forecasting and comparative structural analysis" by Antanina Hryshchuk, Stefan Lessmann, February 2018.

009 "How Sensitive are Tail-related Risk Measures in a Contamination Neighbourhood?" by Wolfgang Karl Härdle, Chengxiu Ling, February 2018.

IRTG 1792, Spandauer Straße 1, D-10178 Berlin
http://irtg1792.hu-berlin.de

This research was supported by the Deutsche Forschungsgemeinschaft through the IRTG 1792.