Uniform Confidence for Pricing Kernels

Wolfgang Karl Härdle
Yarema Okhrin
Weining Wang

Ladislaus von Bortkiewicz
Chair of Statistics
C.A.S.E.-Center for Applied Statistics
and Economics
Humboldt-Universität zu Berlin
University of Augsburg

Motivation

- Arbitrage free market; riskless bond with rate r
- Underlying price process $\{S_t\}$

Price z_t at t of derivative with from a payoff $\psi(S_T)$

$$
\begin{aligned}
z_t &= \int_0^\infty exp(-r\tau)\psi(x)dQ_{S_T}(x) \\
&= \int_0^\infty exp(-r\tau)\psi(x)\frac{q_t(x)}{p_t(x)}dP_{S_T}(x)
\end{aligned}
$$

Figure 1: conditional measure at time of maturity T built opon a path of the stochastic process for underlying asset with information up to time t .

Empirical Pricing Kernel (EPK)

Pricing Kernel (PK) a stochastic discount factor, i.e.

$$\mathcal{K}_{t,\tau}(x) = exp(-r\tau)\frac{q_t(x)}{p_t(x)}$$

EPK is therefore an estimate of PK:

$$\widehat{\mathcal{K}_{t,\tau}(x)} = exp(-r\tau)\frac{\widehat{q_t(x)}}{\widehat{p_t(x)}}$$

The EPK Paradox

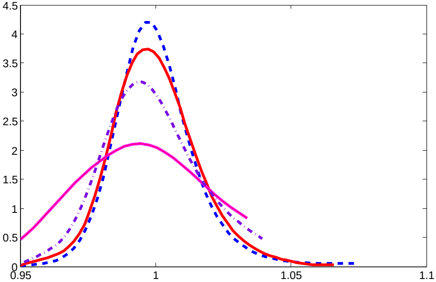


Figure 2: Examples of inter-temporal pricing kernels with maturity 0.00833($3D$) respectively on 17-Jan-2006 (blue), 18-Apr-2006 (red), 16-May-2006 (magenta), 13-June-2006 (black).

The EPK Paradox

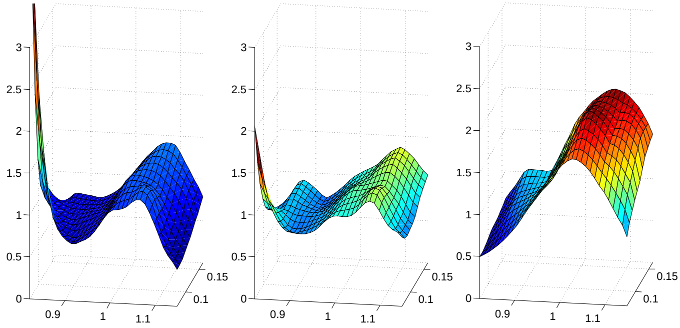
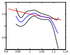


Figure 3: Estimated PK across moneyness and maturity



The EPK Paradox

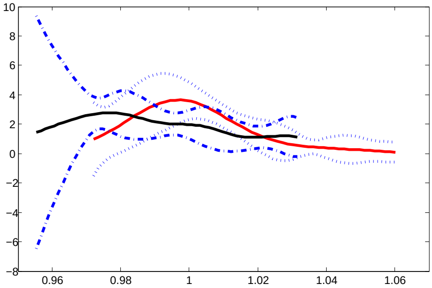


Figure 4: Examples of inter-temporal pricing kernels with various maturities in years: 0.02222 (8D) (red) 0.1(36D) (green) on 12-Jan-2006 and their confidence bands

Aims

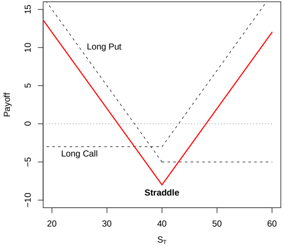
- Nonparametric confidence band to test alternatives
- Check the statistical significance of the EPK puzzle
- Investigate shapes of EPKs: investor preferences
- Understand the dynamics of risk patterns
- Correlate with macro economics

Outline

1. Motivation ✓
2. Uniform Confidence Band
3. Monte-Carlo Study
4. Empirical Data Analysis

European Option

$$\psi(u) = \max(u - K, 0)$$



$$H_t(K,\tau) = \exp(-r\tau) \int \max(u - K, 0) q(u) du$$

Risk Neutral Density (RND) Estimation

The RND may be estimated from option prices, Breeden and Litzenberger (1978):

$$q_t(S_T) = exp(r\tau)\frac{\partial^2 H_t(k,\tau)}{\partial k^2}\Big|_{k=S_T} \tag{1}$$

with call price function $H_t(k,\tau)$.

Aït-Sahalia and Lo (1998) estimate $H_t(k,\tau)$ nonparametrically and differentiate it twice w.r.t. k.

Call prices (x_i, Y_i), fixed τ:

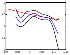
$$Y_i = H(x_i) + \varepsilon_i, i = 1, \dots, n_q$$

Define $L\{y; H(u)\}$ as the conditional density of Y given $K = u$
Local polynomial estimate, $(x \approx u)$:

$$H(u) \approx H(x, u) \stackrel{\text{def}}{=} \sum_{j=0}^{3} H_j(x)(u - x)^j$$

Local likelihood

$$L_{n_q}\{H(x)\} \stackrel{\text{def}}{=} \frac{1}{n_q} \sum_{i=1}^{n_q} K_{h_{n_q}}(x - x_i) \log L\{Y_i; H(x_i, x)\},$$



Local likelihood:

$$-n_q^{-1} \sum_{i=1}^{n_q} K_{h_{n_q}}(x - x_i)\{Y_i - H(x_i, x)\}^2$$

with constant known σ.
Typically $n_q \approx 5000$.

Data

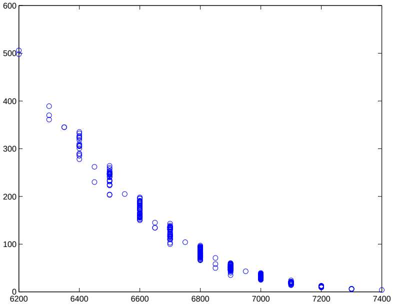


Figure 5: Plot of call prices against strikes k, $n_q = 1000$, $n_p = 500$.

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de

Solution

$$\widehat{\mathbf{H(x)}} \stackrel{\text{def}}{=} \operatorname{argmax}_H L_{n_q}\{H(x)\},$$

where

$$\widehat{\mathbf{H(x)}} = \{\widehat{H_0(x)}, \widehat{H_1(x)}, \widehat{H_2(x)}, \widehat{H_3(x)}\}^\top$$

Estimate for $q_t(x)$

$$\widehat{q_t(x)} \propto 2!\widehat{H_2(x)}$$

Kernel density estimate for $p_t(x)$ is based on historical $\{S_t\}$:

$$\widehat{p_t(x)} = n_p^{-1} \sum_{j=1}^{n_p} K_{h_{n_p}}(x - S_j)$$

Typically $n_p = 500$.

Uniform Convergence

Theorem

Under regularity conditions, for all x in an interval J, we have a.s.,

$$\sup_{x \in J} |\hat{\mathcal{K}}_{t,\tau}(x) - \mathcal{K}_{t,\tau}(x)| = \mathcal{O}[\max\{(n_p h_{n_p} / \log n_p)^{-0.5} + h_{n_p}^2, \\ h_{n_q}^2 + h_{n_q}^{-2}\{n_q h_{n_q} / \log n_q\}^{-0.5}\}]$$

Uniform Confidence Band

Theorem

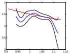
Under regularity conditions,

$$\mathcal{K}_{nt,\tau}(x) \stackrel{\text{def}}{=} n_q^{1/2} {h_{n_q}}^{5/2} \{\widehat{\mathcal{K}_{t,\tau}(x)} - \mathcal{K}_{t,\tau}(x)\} \widehat{\text{Var}}\{\widehat{\mathcal{K}_{t,\tau}(x)}\}^{-1/2}.$$

We have:

$$\mathrm{P}\left\{(-2\log h_{n_q})^{1/2}\left\{\sup_{x\in J}|\mathcal{K}_{nt,\tau}(x)| - c_{nt}\right\} < z\right\} \longrightarrow \exp\{-2\exp(-z)\},$$

where $c_{nt} = (-2\log h_{n_q})^{1/2} + (-2\log h_{n_q})^{-1/2}\{x_\alpha + \log(C/2\pi)\}$



Uniform Confidence Band

Thus, a $(1-\alpha)100\%$ confidence band for pricing kernel $\mathcal{K}_{t,\tau}$ is:

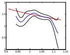
$$[f(x) : \sup_x \{ |\widehat{\mathcal{K}_{t,\tau}(x)} - f(x)| \widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau}(x)})^{-1/2} \} \leq L_\alpha]$$

where

$$L_\alpha = 2!(n_q h_{n_q}^5)^{-1/2} c_{nt}$$

and

$$x_\alpha = -\log\{-1/2 \log(1-\alpha)\}$$



Extension on τ

Let $\mathfrak{x}$ be the possible set of maturities, the extension of our results over τ :

$$[f_{t,\tau}(x) : \sup_{x\in E,\tau\in\mathfrak{x}} \{|\widehat{\mathcal{K}_{t,\tau}(x)} - f_{t,\tau}(x)|\widehat{\text{Var}}(\widehat{\mathcal{K}_{t,\tau}(x)})^{-1/2}\} \leq L_\alpha].$$

In the BS setup, the evolution of bands over time, for fixed τ_1 $(g(\tau_1 - \tau_2) = \mathcal{K}_{t,\tau_1}(x)/\mathcal{K}_{t,\tau_2}(x)))$

$$[f_{t,\tau_2} : \widehat{g}(\tau_1 - \tau_2)\{-L_\alpha \widehat{\text{Var}}(\widehat{\mathcal{K}_{t,\tau_1}(x)})^{1/2} + \widehat{\mathcal{K}_{t,\tau_1}(x)}\} \leq f_{t,\tau_2}(x) \leq \widehat{g}(\tau_1 - \tau_2)\{L_\alpha \widehat{\text{Var}}(\widehat{\mathcal{K}_{t,\tau_1}(x)})^{1/2} + \widehat{\mathcal{K}_{t,\tau_1}(x)}\}],$$

Extension on τ

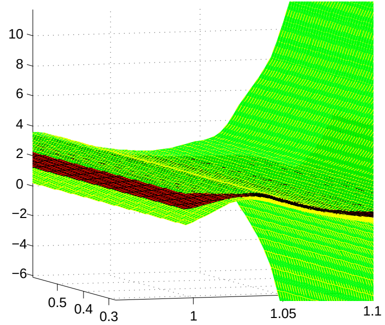


Figure 6: Examples of sheet for pricing kernels in 060228

Bootstrap

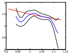
Theorem

Under regularity conditions

$$[f_{t,\tau} : \sup_{x \in E}\{|\widehat{\mathcal{K}_{t,\tau}(x)} - f_{t,\tau}(x)|\widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau}})^{-1/2}\} \leq L^*_\alpha]$$

where the bound L^*_α satisfies

$$\mathrm{P}^*(-\{U_{n_q}(x)^{-1}H_{n_q}^{-1}A^*_{n_q}(x)/B_{t,\tau}(x)^{-1}N(x)^{-1}M^*(x)N(x)^{-1}\}_{3,3} \leq L^*_\alpha) = 1 - \alpha$$



A Monte-Carlo Study

For $q_t(x)$, generate data from BS model, interest rate $r = 0.04$, $S_t = 6500$, $k \in [6200, 7400]$, $\tau = 1\text{M}$, $\varepsilon_i \in \text{U}[0, 6]$, $\sigma = 0.1878$.

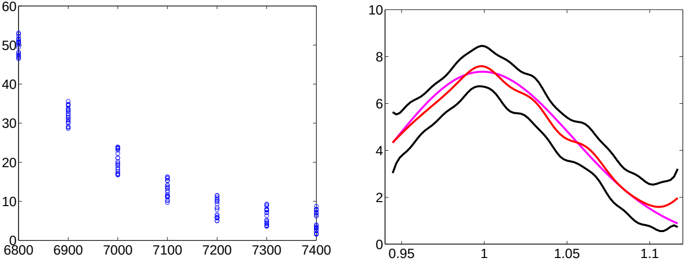


Figure 7: (Left) H against k (Right) Plot of confidence bands (black) , estimated value, the Black Scholes SPD (magenta) of the EPK, $h_{n_q} = 0.085$, $\alpha = 0.05$, $n_q = 300$.

A Monte-Carlo Study

For historical density, simulate data from Geometric Brownian Motion, with $\mu = 0.23$, $\sigma = 0.1878$.

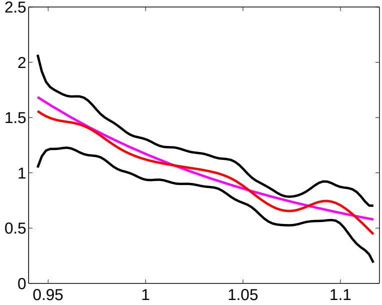
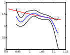


Figure 8: Plot of confidence bands, estimated value, the Black Scholes EPK (magenta), $h_{n_q} = 0.060$, $\alpha = 0.05$, $n_q = 500$, $n_p = 600$.



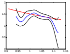
Coverage Probability

Case	$(n=)$300	450	600
$(\tau=)$3	0.9063(2.402)	0.9144(2.204)	0.9233(1.998)
6	0.8964(2.438)	0.9056(2.134)	0.9203(2.069)

Table 1: Cov. prob. (area) of the uniform confidence band for $q_t(x)$ at $\alpha = 5\%$ with $\sigma = 0.1878$, sim $= 500$

Case	$(n=)$300	450	600
3	0.7820(2.5434)	0.7980(2.4978)	0.8020(2.4131)
6	0.8602(2.4987)	0.8749(2.4307)	0.8900(2.4131)

Table 2: Same for EPK at $\alpha = 10\%$



Empirical Data Analysis

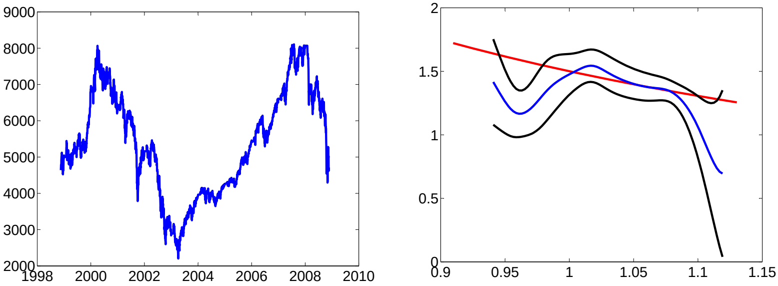


Figure 9: (Left) Plot of DAX index (Right) Plot of confidence bands (black), EPK by Black Scholes fitting, nonparametric EPK, $h_{n_q} = 0.075$, $\alpha = 0.05$, $n_p = 506$, $n_q = 715$.

Empirical Data Analysis

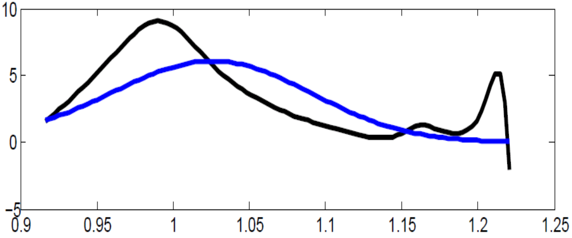
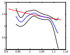


Figure 10: Plot of q and p



Empirical Data Analysis

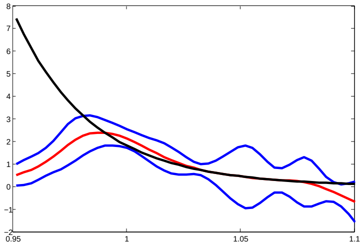
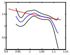


Figure 11: (Left) Plot of confidence bands (blue), EPK by Black Scholes fitting (black), EPK, 2006, July, 24th. (Right) Same for 2006, Aug, 18th.



Economics

Two accuracy measures

- Coverage Probabilities (CP)
- Average Width of Confidence Bands

CP defined via number of grid points (100) containing the BS EPK.

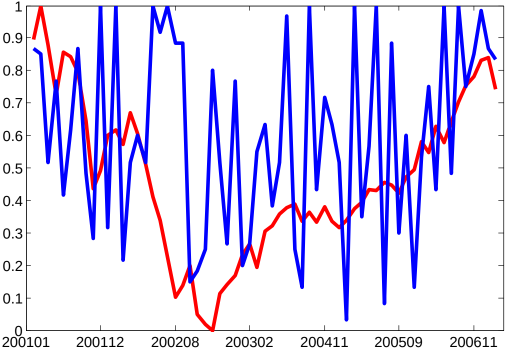


Figure 12: Plot of estimation of the BS EPK covered in band, DAX price (red) $\tau = 2$M.(200001-200006)

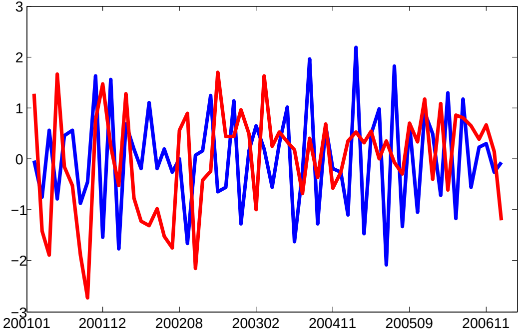
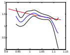


Figure 13: Plot of estimation of the BS EPK covered in band difference(blue), DAX price difference (red) $\tau = 2$M.(200001-200006)



Economics

- CPs become less volatile for bullish market
- High correlation at 3M lag
- DAX returns highly negatively correlated for bearish market
- High positive correlation for bullish DAX

Economics

- Worsening economic conditions, positive amount of DAX returns induces risk hunger
- Bullish markets: positive correlation indicates decreasing risk aversion

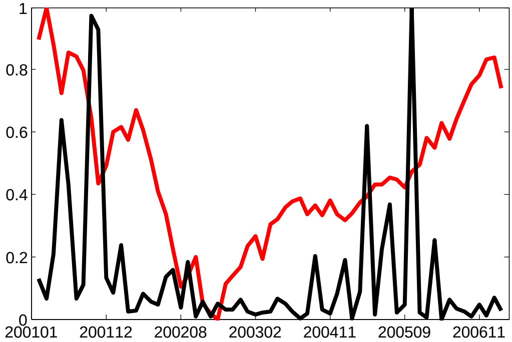


Figure 14: Plot of estimation of area of the bands (blue), DAX price (red) $\tau = 2$M.(200001-200006)

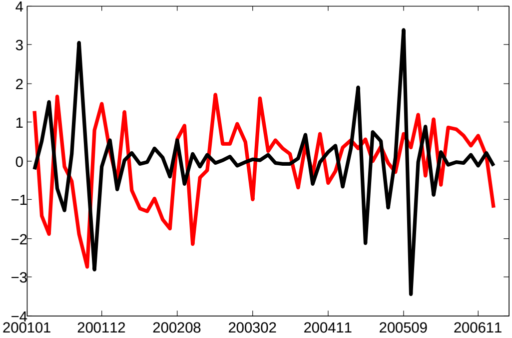
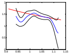


Figure 15: Plot of estimation of area difference (blue), DAX price difference (red) $\tau = 2$M.(200001-200006)

Economics

- For clear bullish or bearish momentum, the volatility of the bands is high
- DAX return strongly negatively correlated with the width difference



Conclusions

- Uniform confidence bands tell us about risk patterns
- Smoothing of EPK is best done via IVS
- Bootstrap does not improve coverage probability significantly
- BS for $\tau = 0.5$M is mostly rejected
- Bootstrap improvement possible for robust smoothers

Uniform Confidence for Pricing Kernels

Wolfgang Karl Härdle
Yarema Okhrin
Weining Wang

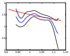
Ladislaus von Bortkiewicz
Chair of Statistics
C.A.S.E.-Center for Applied Statistics
and Economics
Humboldt-Universität zu Berlin
University of Augsburg

Bibliography

Aït-Sahalia, Y., and A. W. Lo.
Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices
Journal of Finance, 1998, 53, 499-547

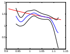
Aït-Sahalia, Y., and A. W. Lo.
Nonparametric Risk Management and Implied Risk Aversion
Journal of Econometrics, 2000, 94, 9-51

Breeden, D. T. and Litzenberger, R. H.
Prices of state-contingent claims implicit in option prices
Journal of Business, 1978, 51, 621-651



Bibliography

- Detlefsen, K., Härdle,W., and Moro,R.
  Empirical Pricing Kernels and Investor Preference
  Mathematical Methods in Economics and Finance, forthcoming
- Engle, R. F. and Rosenberg, J. V.
  Empirical pricing kernels
  Journal of Financial Economics, 2002, 64, 341-372
- Claeskens, G. and Van Keilegom, I.
  Bootstrap Confidence Bands for Regression Curves and Their Derivatives
  The Annals of Statistics, 2003, 31, 1852-1884

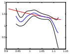


Bibliography

Giacomini, E., Handel, M. and Härdle, W.
Time Dependent Relative Risk Aversion
Risk Assessment: Decisions in Banking and Finance, Bol, G., Rachev, S., Würth, R., Springer, 2008, 15-46

Härdle, W., Krätschmer, V. and Moro, R.
A Microeconomic Explanation of the EPK Paradox
Journal of Financial Econometrics, submitted on 18.09.09

Härdle, W. and Hlavka, Z.
Dynamics of state price densities
Journal of Econometrics, 2009, 150, 1-15

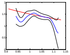


Bibliography

Jackwerth, J.C.
Risk Aversion from Option Prices and Realized Returns
The Review of Financial Studies, 2000, 13, 433-451

Leland, H. E.
Who should buy portfolio insurance?
Journal of Finance, 1980, 32, 581-594

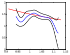
Rookley, C.
Fully exploiting the information content of intra day option quotes: Applications in option pricing and risk management
University of Arizona working paper, 1997



Rookley (1997)

Let C_{it} be the price of the i^{th} option at time t and K_{it} its strike price, and define the rescaled call option $c = C/S_t$ in terms of moneyness $M = S_t/K$ s.t.

$$
\begin{aligned}
c_{it} &= c\{M_{it};\sigma(M_{it})\} = \Phi(d_1) - \frac{e^{-r\tau}\Phi(d_2)}{M_{it}} \\
d_1 &= \frac{\log(M_{it}) + \left\{r_t + \frac{1}{2}\sigma(M_{it})^2\right\}\tau}{\sigma(M_{it})\sqrt{\tau}} \\
d_2 &= d_1 - \sigma(M_{it})\sqrt{\tau}
\end{aligned}
$$



The RND is then

$$q(\cdot) = e^{-r\tau}\frac{\partial^2 C}{\partial K^2} = e^{r\tau} S \frac{\partial^2 c}{\partial K^2}$$

with

$$\frac{\partial^2 c}{\partial K^2} = \frac{\mathrm{d}^2 c}{\mathrm{d}M^2}\left(\frac{M}{K}\right)^2 + 2\frac{\mathrm{d}c}{\mathrm{d}M}\frac{M}{K^2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2 c}{\mathrm{d}M^2} \quad = \quad & \varphi(d_1)\left\{\frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} - d_1\left(\frac{\mathrm{d}d_1}{\mathrm{d}M}\right)^2\right\} \\
& - \frac{e^{-r\tau}\varphi(d_2)}{M}\left\{\frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} - \frac{2}{M}\frac{\mathrm{d}d_2}{\mathrm{d}M} - d_2\left(\frac{\mathrm{d}d_2}{\mathrm{d}M}\right)^2\right\} \\
& - \frac{2e^{-r\tau}\Phi(d_2)}{M^3}
\end{aligned}$$

With $V = \sigma(M)$, $V' = \dfrac{\partial \sigma(M)}{\partial M}$, $V'' = \dfrac{\partial^2 \sigma(M)}{\partial M^2}$

$$\begin{aligned}
\frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} &= -\frac{1}{M * V(M)\sqrt{\tau}}\left\{\frac{1}{M} + \frac{V'(M)}{V(M)}\right\} \\
&\quad + V''(M)\left\{\frac{\sqrt{\tau}}{2} - \frac{\log(M) + r\tau}{V(M)^2\sqrt{\tau}}\right\} \\
&\quad + V'(M)\left\{2V'(M)\frac{\log(M) + r\tau}{V(M)^3\sqrt{\tau}}\right. \\
&\quad \left. - \frac{1}{M * V(M)^2\sqrt{\tau}}\right\}
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} \quad = \quad & -\frac{1}{M * V(M)\sqrt{\tau}}\left\{\frac{1}{M} + \frac{V'(M)}{V(M)}\right\} \\
& - V''(M)\left\{\frac{\sqrt{\tau}}{2} + \frac{\log(M) + r\tau}{V(M)^2\sqrt{\tau}}\right\} \\
& + V'(M)\Big\{ 2V'(M)\frac{\log(M) + r\tau}{V(M)^3\sqrt{\tau}} \\
& - \frac{1}{M * V(M)^2\sqrt{\tau}}\Big\}
\end{aligned}
$$