Testing Monotonicity of Pricing Kernels

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Is There a Bump?

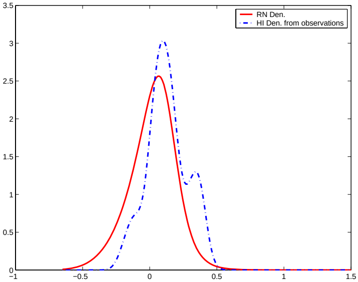


Figure 1: Neutral and historical densities for 1/2 year returns in 2000

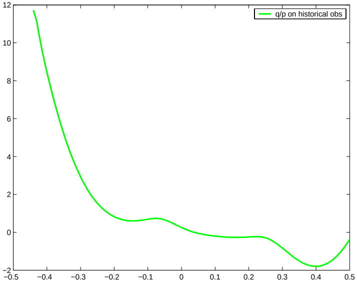


Figure 2: Corresponding Empirical Pricing Kernel (EPK), 2000

Basic Notions

R_t - half a year DAX returns defined as $R_t = \frac{S_t - S_{t-126}}{S_{t-126}}$ where S_t is observed DAX value at time t

p is the unknown density of the returns R_t

q is the risk neutral density estimated from implied volatility surfaces over the observed option prices with maturities of half year.

Due to sufficient number of observations on options, q can be precisely estimated and is considered to be known.

Basic Notions

Let us define the pricing kernel K as

$$K(R_t) = \frac{q(R_t)}{p(R_t)}$$

Problem: Test monotonicity of pricing kernel $K(x) = \frac{q(x)}{p(x)}$ given that p is unknown

Monotonicity of Pricing Kernel

Monotone decreasing pricing kernel corresponds to classical risk-averse utility function.

Non-monotone pricing kernel leads to non-concave utility function and contradicts to Friedman's concept of risk averse behavior.

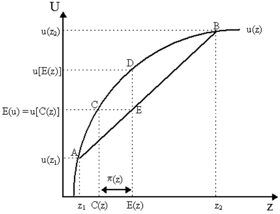


Figure 3: Risk-Aversion and Certainty Equivalence

Outline of the Talk

1. Motivation ✓
2. Problem Setup
3. Problem Simplification
4. Values
5. Applications and Simulations

Problem Setup

For increasing sequence of returns $R_{(k)}$ such that

$$R_{(1)} \leq R_{(2)}, \ldots, \leq R_{(n)}$$

we test if there exists any interval I, J such that the sequence

$$K_{(k)} = \frac{q(R_{(k)})}{p(R_{(k)})}, I \leq k \leq J$$

is monotone decreasing having only $q(R_t)$ as a known density

Test Outline

1. Reduce our problem to a simpler problem using Pyke's order statistics results

2. Construct likelihood ratio test for a nested (reduced) model

3. Compute corresponding test statistics

4. Simulate test distribution via Monte Carlo simulations in order to find critical values

5. Take decision about monotonicity of pricing kernel K

Order Statistics

Consider $U_1, \ldots, U_n$ be *i.i.d* with a uniform distribution on $[0,1]$. For the order statistics

$$U_{(1)} \leq U_{(2)}, \ldots, \leq U_{(n)}$$

define *uniform spacings* S_k as

$$S_1 = U_{(1)} \text{ and } S_k = U_{(k)} - U_{(k-1)}$$

Pyke's Theorem

Theorem *Let* $U_1, \ldots, U_n$ *be i.i.d with a uniform distribution on* $[0, 1]$ *and* $e_1, \ldots, e_n$ *be i.i.d. standard exponentially distributed random variables. Then for uniform spacings* S_k

$$\mathcal{L}\{S_k, 1 \leq k \leq n\} = \mathcal{L}\left\{\frac{e_k}{\sum_{i=1}^{n} e_k}, 1 \leq k \leq n\right\}$$

Using the fact that $E(e_k) = 1$ we obtain the following result

$$n\left\{U_{(k)} - U_{(k-1)}\right\} = n \cdot S_k \approx e_k \tag{1}$$

Problem Simplification

Let $P(x)$ be a cumulative distribution function associated with pdf $p(x)$ such that

$$P(x) = \int_{-\infty}^{x} p(u)du$$

Using Taylor approximation

$$P(R_{(k+1)}) = P(R_{(k)}) + P'(R_{(k)}) \cdot (R_{(k+1)} - R_{(k)})$$

we can write

$$U_{(k+1)} - U_{(k)} = P(R_{(k+1)}) - P(R_{(k)}) \approx p(R_{(k)}) \cdot \left(R_{(k+1)} - R_{(k)}\right) \tag{2}$$

Problem Simplification

Combining (1) and (2) we obtain the main simplification result

$$n \cdot q(R_{(k)}) \cdot \left(R_{(k+1)} - R_{(k)}\right) \approx \frac{q(R_{(k)})}{p(R_{(k)})} \cdot e_k = K_k \cdot e_k \qquad (3)$$

Taking $n \cdot q(R_{(k)}) \cdot \left(R_{(k+1)} - R_{(k)}\right) = Z(R_{(k)}) = Z_k$ our problem is reduced to the following

Check monotonicity of $K(R_{(k)}) = K_k$ using

$$Z_k = K_k \cdot e_k$$

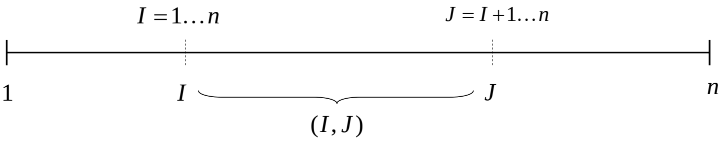
Testing Hypothesis

Let $A(I,J)$ be set of all positive decreasing sequences

$$A(I,J) = \{a_k \geq 0 : a_k \geq a_{k+1}, I \leq k \leq J\}$$

Hypothesis H_0**:** $K \subset A(I,J)$ and pricing kernel $K(R_k)$ is monotone decreasing function

Hypothesis H_1**:** K is not monotone decreasing function



Likelihood Ratio Test

The likelihood ratio monotonicity test is defined by the function

$$\phi(Z) = \mathbf{1}\left\{ \frac{\max_{K \subset A(I,J)} log \left\{ p(Z,K) \right\}}{\max_K log \left\{ p(Z,K) \right\}} - h_\alpha(I,J) \geq 0 \right\} \qquad (4)$$

where $h_\alpha(I,J)$ is our critical surface with significance level α

H_0 is rejected if $\phi(Z) \neq 1$ and accepted if $\phi(Z) = 1$

Computation of ML function

Using the results from equation (3) that $Z_k = K_k \cdot e_k$ we derive maximum likelihood function

$$log\left\{p(Z,K)\right\} = -\sum_{k=I}^{J} \frac{Z_k}{K_k} - \sum_{k=I}^{J} log(K_k) \tag{5}$$

which gives us analytical result for $\max_K log\left\{p(Z,K)\right\}$

$$\max_K log\left\{p(Z,K)\right\} = -n - \sum_{k=1}^{n} log(Z_k)$$

Computation of ML function

Computation of $\max_{K \subset A(I,J)} log \left\{ p(Z,K) \right\}$ is performed with **Newton-Raphson method with the projection on decreasing sequence** $A(I,J)$.

The main idea of the method is to find the maximum likelihood over all possible monotone decreasing sequences by interative optimization via the Newton Raphson algorithm.

The result of 'best' decreasing sequences can be achieved through Isotonic regression combined with Newton Raphson opimization algorithm.

Isotonic Regression and Newton Raphson Algorithm

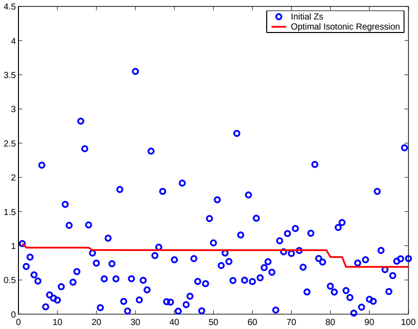


Figure 4: Isotonic Regression combined with Newton Rapshon algorithm

Critical Values

Generate 'the worst' non-increasing case of the sequence $K_{(k)}$ as a constant:

$$K_{(1)} = K_{(2)} = \ldots = K_{(n)} = 1$$

Then using the result that $Z_k = K_k \cdot e_k$ we generate $Z_k \approx exp(1)$ as an $i.i.d$ standard exponential random variable

Critical Values

Define $\xi(I,J)$ as a test statistics over simulated Z_k:

$$\xi(I,J) = \frac{\max_{K \subset A(I,J)} log\{p(Z,K)\}}{\max_K log\{p(Z,K)\}}$$

Define mean $M(I,J)$ and variance $V^2(I,J)$ of $\xi(I,J)$:

$$M(I,J) = E_0 \xi(I,J)$$
$$V^2(I,J) = E_0 \left[\xi^2(I,J) - E_0 \xi(I,J)\right]^2$$

Parameters $M(I,J)$ and $V(I,J)$ are calculated by simulations.

Critical Values

Critical value t_α where α is a significance level is calculated as a root to equation

$$P_0 \left\{ \max_{I=1,n} \max_{J=I+1,n} \{\xi(I,J) - M(I,J) - t_\alpha V(I,J)\} \geq 0 \right\} = \alpha \quad (6)$$

over Monte-Carlo simulations.

Equation (6) gives us a corresponding critical surface $h_\alpha(I,J)$

$$h_\alpha(I,J) = M(I,J) + t_\alpha \cdot V(I,J)$$

Test Summary

1. Compute $Z(R_{(k)}) = n \cdot q(R_{(k)}) \cdot \left\{R_{(k+1)} - R_{(k)}\right\}$
2. Compute test statistics

$$\xi(I,J) = \frac{\max_{K \subset A(I,J)} log\{p(Z,K)\}}{\max_K log\{p(Z,K)\}}$$

3. Take decision: if

$$\max_{I=1,n} \max_{J=I+1,n} \{\xi(I,J) - M(I,J) - t_\alpha V(I,J)\} \geq 0$$

then $K(\cdot)$ is a non-monotone function

Real Z_k

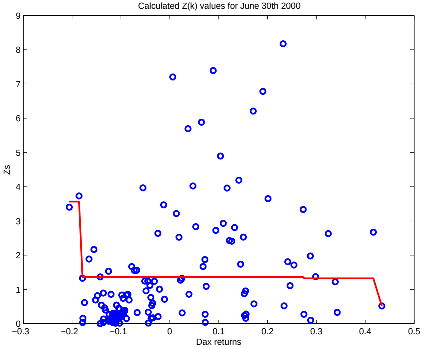


Figure 5: Calculated Z_k and optimized isotonic regression over DAX returns data in 2002

Simulated surfaces $M(I, J)$ and $V(I, J)$

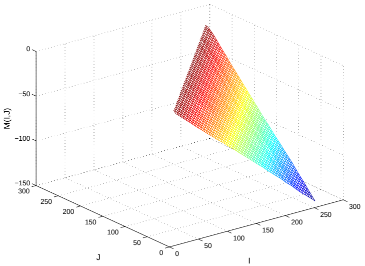


Figure 6: Simulated surface $M(I, J)$ with Monte-Carlo method, n=255

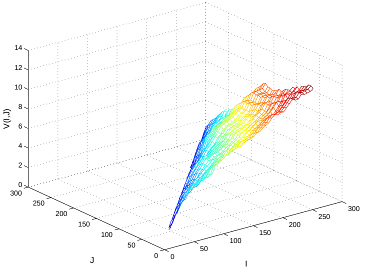


Figure 7: Simulated surface $V(I, J)$ with Monte-Carlo method, , n=255

Simulated t_α

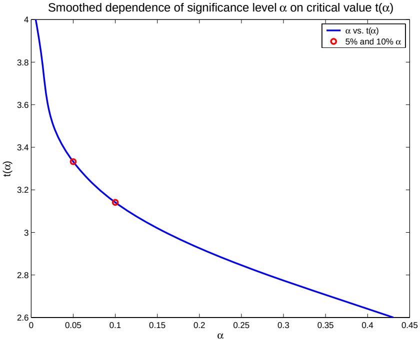


Figure 8: Critical values t_α for different significance levels α

Critical surfaces $h_\alpha(I, J)$

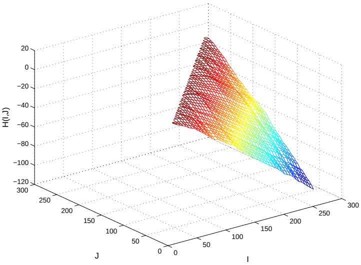


Figure 9: Critical surfaces $h_\alpha(I, J)$ for significance level $\alpha = 5\%$

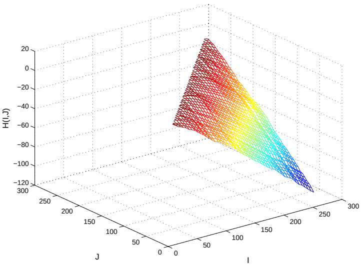


Figure 10: Critical surfaces $h_\alpha(I, J)$ for significance level $\alpha = 10\%$

Obtained results for EPK in 2002

α significance Level	**5%**	**10%**	**20%**
t(α) critical value	3.3325	3.1409	2.9259
Test Value $\max_{I=1,n} \max_{J=I+1,n} \left\{ \xi(I,J) - M(I,J) - t(\alpha) V(I,J) \right\}$	-0.9474	-0.8839	-0.8127
Hypothesis (H0/H1)	H0	H0	H0

Figure 11: Summary of results

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