Partial Linear Quantile Regression and Bootstrap Confidence Bands

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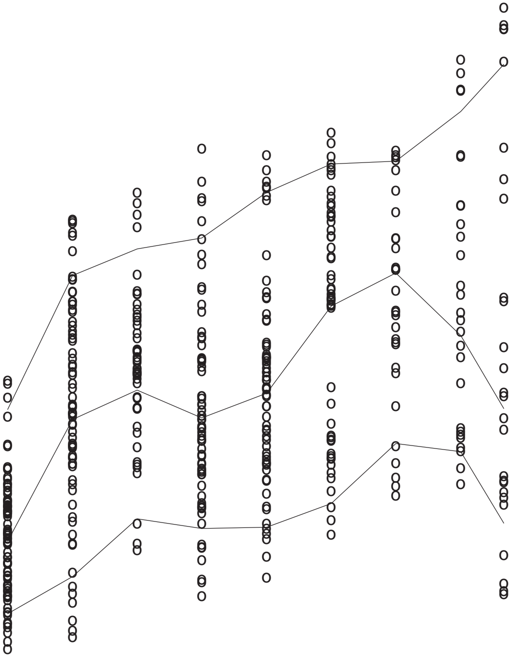
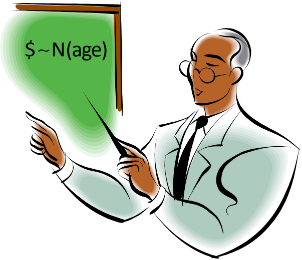
Risky to see only part(s) of the truth!





- log(Salary) ∼ Years
- “The rich got richer and the poor got poorer!”
- Yu et al. (2003)

Quantile Regression

- QR: conditional behavior of a response Y
- Median regression $=$ mean regression (symmetric)
- "Gradually developing into a comprehensive strategy for completing the regression prediction", Koenker & Hallock (2001)

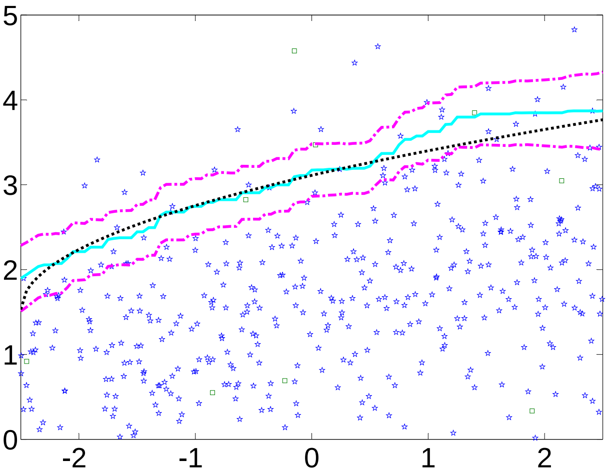


Figure 1: The 0.9-quantile curve, the 0.9-quantile smoother with $h_{0.9} = 1.25$ and 95% confidence bands. QR1

Example

- Financial Market & Econometrics
  - VaR (Value at Risk) tool to measure risk, Lauridsen (2000)
  - Detect conditional heteroscedasticity, Koenker & Bassett (1982)
- Labor Market
  - Analyse income of football players w.r.t. different ages, years, and countries, etc
  - Investigate discrimination effects, Buchinsky (1995)

$$\begin{aligned}\log(\text{Income}) &= A(\text{year, age, etc}) \\ &\quad +\beta\, B(\text{education, gender, nationality, union status, etc}) + \varepsilon\end{aligned}$$

  - Inequality analysis
  - ...

Quantile Regression

- $l(x) = F^{-1}_{Y|x}(p)$ p-quantile regression curve
- $l(x) =$ linear (parametric) form, Koenker & Bassett (1978)
- $l_h(x)$ quantile-smoother

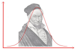
How to decide between functional forms? (global variability of the estimate, peak or valley really a feature?)

Theorem (Härdle and Song (2009))

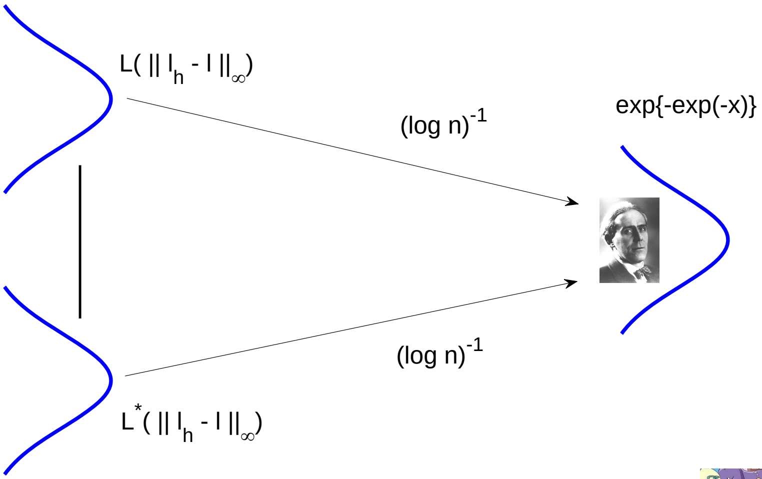
An approximate $(1-\alpha)\times 100\%$ *confidence band over* $[0,1]$ *is*

$$\begin{aligned} l_h(t) \quad \pm \quad & (nh)^{-1/2}\{p(1-p)/\hat{f}_X(t)\}^{1/2}\hat{f}^{-1}\{l(t)|t\} \\ & \times\{d_n + c(\alpha)(2\delta\log n)^{-1/2}\}\cdot\{\lambda(K)\}^{1/2}, \end{aligned} \qquad (1)$$

where $c(\alpha)=\log 2-\log|\log(1-\alpha)|$ *and* $\hat{f}_X(t)$, $\hat{f}\{l(t)|t\}$ *are consistent estimates for* $f_X(t)$, $f\{l(t)|t\}$.

Emil Julius Gumbel on BBI: 

Challenges



Opportunities

- "Hungarian machine gun", $x \in \mathbb{R}^1$ (KMT)
  Tool to prove asymptotic bands
- Extend this to $x \in \mathbb{R}^d$ and improve band precision?
  - Hall (1991): bootstrap can beat it! (density)
  - Hahn (1995): consistency of bootstraping CDF
  - Horowitz (1998): bootstrap (pointwise) for median
  - PLM: Green & Yandell (1985), Denby (1986), Speckman (1988) and Robinson (1988)
  - Variable selection for QR: Liang and Li (2009)

Outline

1. Motivation ✓
2. Bootstrap Confidence Bands
3. Bootstrap Confidence Bands in PLMs
4. Monte Carlo Study
5. Labour Market Applications

Quantile Regression

- $\{(X_i, Y_i)\}_{i=1}^n$ i.i.d. rv's, $x \in J^* = (a, b)$ for some $0 < a < b < 1$, $y \in \mathbb{R}$
- Suppose $Y_i = l(X_i) + \varepsilon_i$, $\varepsilon_i \sim F(\cdot|X_i)$ with $F(0|X_i) = p$. Both l & F are smooth.
- Estimator $l_h(\cdot)$: the solution of

$$\frac{\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i < l_h(x)\}}{\sum_{i=1}^n K_h(x - X_i)} < p \leq \frac{\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i \leq l_h(x)\}}{\sum_{i=1}^n K_h(x - X_i)}$$

- S_n: any slowly varying function (e.g., $S_n^2 = S_n$ is valid...).

- Local rate of convergence of l_h
  $\delta_n = h^2 + (nh)^{-1/2} = \mathcal{O}(n^{-2/5})$ with $h_n = \mathcal{O}(n^{-1/5})$
- Auxiliary estimate l_g with larger bandwidth $g_n = h_n n^{\zeta}$ (ζ: $4/45$)
- $\hat{F}(\cdot|X_i) = \frac{\sum_{j=1}^{n} K_h(X_j - X_i)\mathbf{1}\{Y_j - l_h(X_i) \le \cdot\}}{\sum_{j=1}^{n} K_h(X_j - X_i)}$

Check Function

$$\rho_p(u) = pu\mathbf{1}\{u \in (0,\infty)\} - (1-p)u\mathbf{1}\{u \in (-\infty,0)\}$$

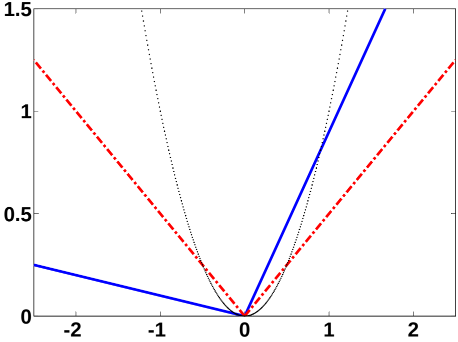


Figure 2: Check function for p=0.9, p=0.5 and weight function in conditional mean regression

The Quantile Curve

$$\rho_p(u) = pu\mathbf{1}\{u \in (0,\infty)\} - (1-p)u\mathbf{1}\{u \in (-\infty,0)\}$$

- $$l(x) = \arg\min_{\theta} \mathsf{E}\{\rho_p(Y-\theta)|X=x\}$$

- $$l_h(x) = \arg\min_{\theta} n^{-1} \sum_{i=1}^{n} \rho_p(Y_i-\theta)K_h(x-X_i)$$

  where $K_h(u) = h^{-1}K(u/h)$ is a kernel (symmetric density function with compact support) with bandwidth h

Weight Function

$$\psi(u) = p - \mathbf{1}\{u \in (-\infty, 0)\}$$

$l_h(x)$ and $l(x)$: treated as a zero of $\widetilde{H}_n\{l_h(x), x\}$ and $\widetilde{H}\{l(x), x\}$

where:

$$\widetilde{H}_n\{l_h(x), x\} = 0 : \qquad \widetilde{H}_n(\theta, x) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^{n} K_h(x - X_i)\psi(Y_i - \theta)$$

$$\widetilde{H}\{l(x), x\} = 0 : \qquad \widetilde{H}(\theta, x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x, y)\psi(y - \theta)dy$$

$\hat{F}$ Approximation Performance around 0

Lemma

[Franke and Mwita (2003), p14] If assumptions (A1, A2, A4) hold, then for any small enough (positive) $\varepsilon \longrightarrow 0$,

$$\sup_{|t|<\varepsilon, i=1,\dots,n, X_i \in J^*} |\hat{F}_i(t) - F(t|X_i)| = \mathcal{O}_p(S_n \delta_n \varepsilon^{1/2} + \varepsilon^2). \qquad (2)$$

The Bootstrap Couple

- $U_1, \ldots, U_n$: i.i.d. uniform $[0, 1]$ rv's
- Bootstrap sample

$$Y_i^* = l_g(X_i) + \hat{F}_i^{-1}(U_i), \quad i = 1, \ldots, n$$

- Couple with the true conditional distribution:

$$Y_i^{\#} = l(X_i) + F^{-1}(U_i|X_i), \quad i = 1, \ldots, n.$$

Given $X_1, \ldots, X_n$: $Y_1, \ldots, Y_n$ and $Y_1^{\#}, \ldots, Y_n^{\#}$ are equally distributed.

A Very Close Couple

$$Y_i^* = l_g(X_i) + \hat{F}_i^{-1}(U_i), \quad i = 1, \ldots, n$$

$$Y_i^{\#} = l(X_i) + F^{-1}(U_i|X_i), \quad i = 1, \ldots, n.$$

Values of $Y_i^{\#}$ and Y_i^* are meaningful only if $|U_i - p| < S_n\delta_n$.
By the inverse function theorem around p, we have

$$\max_{i:|Y_i^{\#} - l(X_i)| < S_n\delta_n} |Y_i^{\#} - l(X_i) - Y_i^* + l_g(X_i)| = \mathcal{O}_p\{S_n\delta_n^{3/2}\}.$$

How Close?

- $q_{hi}(Y_1, \ldots, Y_n) \stackrel{\text{def}}{=} l_h(X_i)$ for data set $\{(X_i, Y_i)\}_{i=1}^n$
- Assumption A3 gives:

$$\max_{|X_i - x_j| < ch} |l_g(X_i) - l_g(X_j) - l(X_i) + l(X_j)| = \mathcal{O}_p(\delta_n)$$

- l_h^* and $l_h^\#$: local bootstrap quantile and its coupled sample analogue. Then

$$l_h^*(X_i) - l_g(X_i) = q_{hi}[\{Y_j^* - l_g(X_j) + l_g(X_j) - l_g(X_i)\})_{j=1}^n]$$

$$l_h^\#(X_i) - l(X_i) = q_{hi}[\{Y_j^\# - l(X_j) + l(X_j) - l(X_i)\}_{j=1}^n]$$

Thus

$$\max_i |l_h^*(X_i) - l_g(X_i) - l_h^\#(X_i) - l(X_i)| = \mathcal{O}_p(\delta_n).$$

Bootstrapping Approximation Rate

Theorem

If assumptions (A1)–(A3) hold, then

$$\sup_{x \in J^*} |l_h^*(x) - l_g(x) - l_h^{\#}(x) - l(x)| = \mathcal{O}_p(\delta_n) = \mathcal{O}_p(n^{-2/5}).$$

- Bootstrap improves the rate of convergence.

Why Oversmoothing?

- To handle the bias (closer). Tuning parameter: g
- Härdle and Marron (1991), let

$$
\begin{aligned}
b_h(x) &\stackrel{\text{def}}{=} \mathsf{E}\, l_h^{\#}(x) - l(x) \\
\hat{b}_{h,g}(x) &\stackrel{\text{def}}{=} \mathsf{E}^*\, l_h^*(x) - l_g(x)
\end{aligned}
$$

- Investigate MSE $\mathsf{E}\left[\left\{\hat{b}_{h,g}(x) - b_h(x)\right\}^2 | X_1, \ldots, X_n\right]$.
  How fast it converges to 0?

Oversmoothing

Theorem

Under some assumptions, for any $x \in J^*$

$$\mathsf{E}\left[\left\{\hat{b}_{h,g}(x) - b_h(x)\right\}^2 |X_1, \ldots, X_n\right] \sim h^4\{\mathcal{O}_p(g^4) + \mathcal{O}_p(n^{-1}g^{-5})\}$$

in the sense that the ratio between the RHS and the LHS tends in probability to 1 for some constants $C_1,\ C_2$.

To minimize MSE, $g = \mathcal{O}(n^{-1/9})$, $g \gg h$, where $h = \mathcal{O}(n^{-1/5})$

The Multivariate Case

- $x = (u, v)^\top \in \mathbb{R}^d$, $v \in \mathbb{R}$:

$$\tilde{l}(x) = u^\top \beta + l(v)$$

- Estimation idea: ANOVA, approximately linear form (locally)
- Partition $[0, 1]$ (for v) in a_n intervals I_{ni} & regard $l(v)$ as a constant item inside I_{ni}.

Two Stage Estimation Procedure

- Linear quantile regression inside each I_{ni} + Weighted mean yields $\hat{\beta}$:

$$\hat{\beta} = \arg\min_{\beta} \min_{l_1,\dots,l_{a_n}} \sum_{i=1}^{n} \psi\{Y_i - \beta^\top U_i - \sum_{j=1}^{a_n} l_j \mathbf{1}(V_i \in I_{ni})\}$$

- Smooth quantile estimate $\hat{l}_h(v)$ from $(V_i, Y_i - U_i^\top \hat{\beta})_{i=1}^{n}$.

Theorem

$\exists$ *positive definite matrices* $D_n,\ C_n$, *s.t.*

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{\mathcal{L}} N\{0, p(1-p)D_n^{-1} C_n D_n^{-1}\} \text{ as } n \to \infty.$$

Uniform Consistency of $\hat{l}_h(v)$

Lemma

Under assumptions (A7) & (A8), we have a.s. as $n \to \infty$

$$\sup_{v \in J^*} |\hat{l}_h(v) - l(v)| \leq C_5 \max\{(nh/\log n)^{-1/2}, h^{\tilde{\alpha}}\} \qquad (3)$$

with another constant C_5 not depending on n. If additionally $\tilde{\alpha} \geqslant \{\log(\sqrt{\log n}) - \log(\sqrt{nh})\}/\log h$, (3) can be further simplified to:

$$\sup_{v \in J^*} |\hat{l}_h(v) - l(v)| \leq C_5\{(nh/\log n)^{-1/2}\}.$$

Multidimensional Uniform Confidence Bands

- Estimation error for parametric part: $\mathcal{O}_p(n^{-1/2})$.
- Bootstrapping approximation error for nonparametric part: $\mathcal{O}_p(n^{-2/5})$, dominating!

Corollary

Under the assumptions (A1) - (A8), an approximate $(1-\alpha)\times 100\%$ confidence band over $\mathbb{R}^{d-1}\times[0,1]$ is

$$u^\top \hat{\beta} + l_h(v) \quad \pm \quad \left[\hat{f}\{l(x)|x\}\sqrt{\hat{f}_X(x)}\right]^{-1} d_\alpha^*,$$

where d_α^* is based on the bootstrap sample (specify later).

How to Bootstrap?

1) Simulate $\{(X_i, Y_i)\}_{i=1}^n, n = 1000$ w.r.t. $f(x, y)$.

$$f(x,y) = f_{y|x}(y - \sin x)\mathbf{1}(x \in [0,1]), \tag{4}$$

where $f_{y|x}(x)$ is the pdf of $\mathrm{N}(0, x)$.

2) Compute $l_h(x)$ of $Y_1, \ldots, Y_n$ and residuals $\hat{\varepsilon}_i = Y_i - l_h(X_i),\ i = 1, \ldots, n$.
If we choose $p = 0.9$, then $\Phi^{-1}(p) = 1.2816$,
$l(x) = \sin(x) + 1.2816\sqrt{x}$ and the bandwidth is $h = 0.05$.

3) Compute the conditional edf $F_{n|x}$:

$$F_n(t|x) = \frac{\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{\hat{\varepsilon}_i \leqslant t\}}{\sum_{i=1}^n K_h(x - X_i)}$$

with the quartic kernel

$$K(u) = \frac{15}{16}(1 - u^2)^2, \quad (|u| \leqslant 1).$$

4) Generate rv $\varepsilon^*_{i,b} \sim F_{n|x}$, $b = 1, \dots, B$ and construct the bootstrap sample $Y^*_{i,b}, i = 1, \dots, n, b = 1, \dots, B$ as follows:

$$Y^*_{i,b} = l_g(X_i) + \varepsilon^*_{i,b},$$

with $g = 0.2$.

5) For each bootstrap sample $\{(X_i, Y^*_{i,b})\}_{i=1}^n$, compute l^*_h and the random variable

$$d_b \stackrel{\text{def}}{=} \sup_{X \in J^*} \left[\hat{f}\{l(x)|x\} \sqrt{\hat{f}_X(x)} |l^*_h(x) - l_g(x)| \right]. \qquad (5)$$

6) Calculate the $(1-\alpha)$ quantile d^*_α of $d_1, \dots, d_B$.

7) Construct the bootstrap uniform confidence band centered around $l_h(x)$, i.e. $l_h(x) \pm \left[\hat{f}\{l(x)|x\} \sqrt{\hat{f}_X(x)} \right]^{-1} d^*_\alpha$.

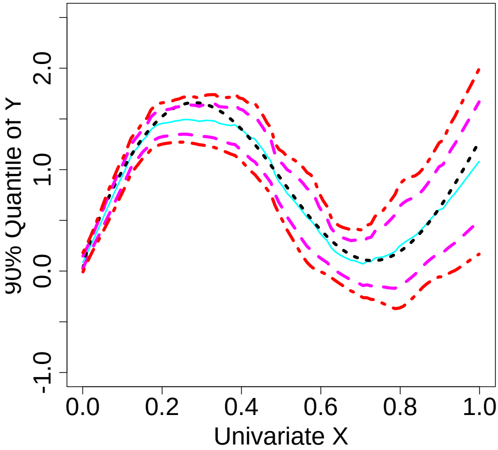


Figure 3: The real 0.9 quantile curve, 0.9 quantile estimate with corresponding 95% uniform confidence band from asymptotic theory and confidence band from bootstrapping.

Convergence Rate (n small)

n	Cov. Prob.	Area
50	0.144 (0.642)	0.58 (1.01)
100	0.178 (0.742)	0.42 (0.58)
200	0.244 (0.862)	0.31 (0.36)

Table 1: Simulated coverage probabilities & areas of nominal asymptotic (bootstrap) 95% confidence bands with 500 repetition.

- For small n, bootstrap's » asymptotic's & not sacrifice much on the band's width
- To achieve same cov. prob., quantile regression usually need more observations than mean regression
- Use larger bandwidth on both X & Y ($\hat{f}^{-1}\{l(x)|x\}$)

PLM QR

- Bivariate data $\{(U_i, V_i, Y_i)\}_{i=1}^{n}, n = 8000$ with:

$$y = 2u + v^2 + \varepsilon - \Phi(p) \tag{6}$$

  where $u \in [0, 2],\ v \in [0, 1]$ and ε is the standard normal rv.

- The real 0.9-quantile curve $\tilde{l}(x) = 2u + v^2$.
- $h = 0.2$ & $g = 0.7$. For the following specific set of random variables, $a_n = 20,\ \hat{\beta} = 2.016758$

of Partitions?

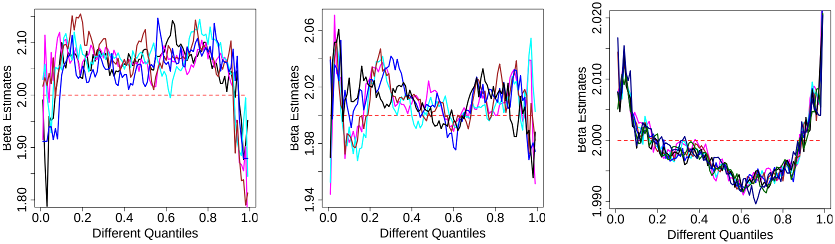


Figure 4: $\hat{\beta}$ with respect to different p for different # of observations, i.e. $n = 1000$, $n = 8000$, $n = 261148$.

a_n	$n = 1000$	$n = 8000$	$n = 261148$
$n^{1/3}/8$			$3.6 * 10^{-3}$
$n^{1/3}/4$	$5.4 * 10^{-1}$	$4.0 * 10^{-2}$	$3.3 * 10^{-3}$
$n^{1/3}/2$	$6.1 * 10^{-1}$	$3.5 * 10^{-2}$	$3.2 * 10^{-3}$
$n^{1/3}$	$6.2 * 10^{-1}$	$3.6 * 10^{-2}$	$3.1 * 10^{-3}$
$n^{1/3} \cdot 2$	$8.0 * 10^{-1}$	$3.9 * 10^{-2}$	$2.9 * 10^{-3}$
$n^{1/3} \cdot 4$	$4.9 * 10^{-1}$	$3.6 * 10^{-2}$	$2.8 * 10^{-3}$
$n^{1/3} \cdot 8$			$3.4 * 10^{-3}$

Table 2: SSE of $\hat{\beta}$ with respect to a_n for different numbers of observations.

- Suggest $a_n = n^{1/3}$ (cost / performance)

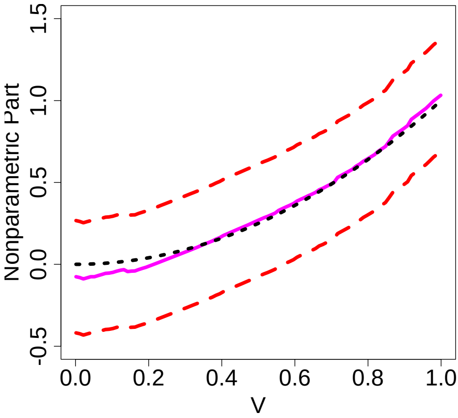


Figure 5: Nonparametric part smoothing, real 0.9 quantile curve with respect to v, 0.9 quantile smoother with corresponding 95% bootstrap uniform confidence band.

Labor Market Application

- How income depends on age w.r.t. different education levels?
- Relation: $\log(\text{Wage}) \sim \beta \cdot \text{Education} + l(\text{Age})$
- Administrative data from the German National Pension Office
- Male, born 1939 $\sim$ 1942, sample 25 - 59, full-time, begin receiving a pension in 2004 $\sim$ 2005

- Education categories: “no answer", “low education”, “apprenticeship” and “university”
- Normal impression:

$$
\begin{aligned}
& \mathsf{E}(y|v, u = \text{Low education}) \\
< \; & \mathsf{E}(y|v, u = \text{Apprenticeship}) \\
< \; & \mathsf{E}(y|v, u = \text{University})
\end{aligned}
$$

Box Plot

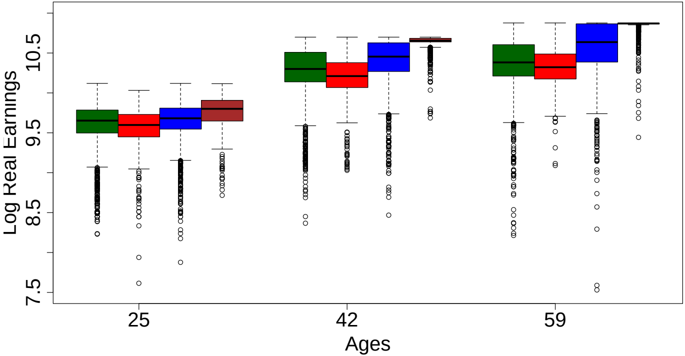


Figure 6: Boxplots for "no answer", "low education", "apprenticeship" & "university" groups corresponding to $v = 0, 0.5, 1$.

- Drop "no answer" group & $n = 175760$ observations
- 1 "low education", 2 "apprenticeship" and 3 "university"
- $175760^{1/3}/2 = 28$ partitions
- Quartic kernel, $h = 0.018$ (after rescaling)

β Estimates

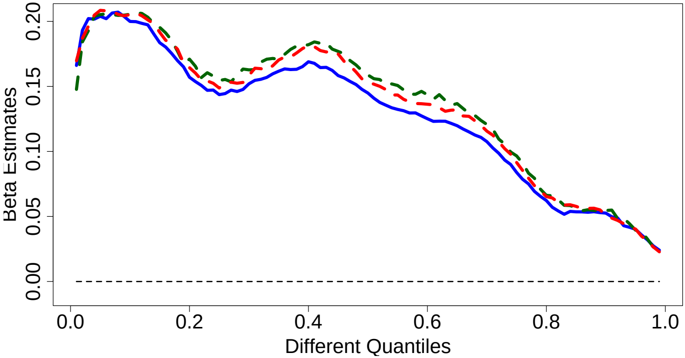


Figure 7: $\hat{\beta}$ corresponding to 8, 16, 32 partitions.

Low Income - Significant

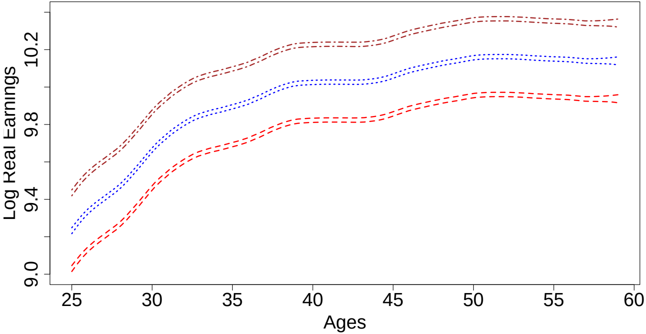


Figure 8: 95% uniform confidence bands for 0.05-quantile smoothers with 3 different education levels "low education", "apprenticeship" & "university".

Median Income - Significant

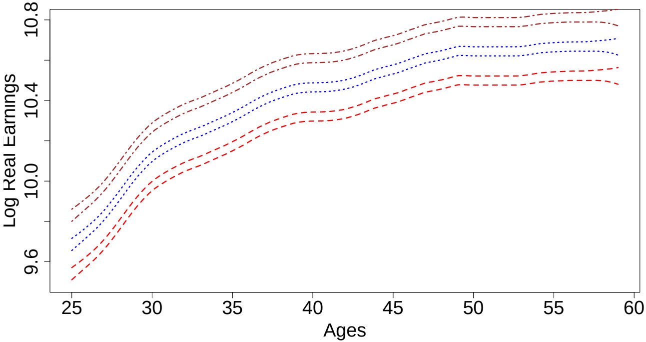


Figure 9: 95% uniform confidence bands for 0.50-quantile smoothers with 3 different education levels "low education", "apprenticeship" & "university".

High Income - Not Significant

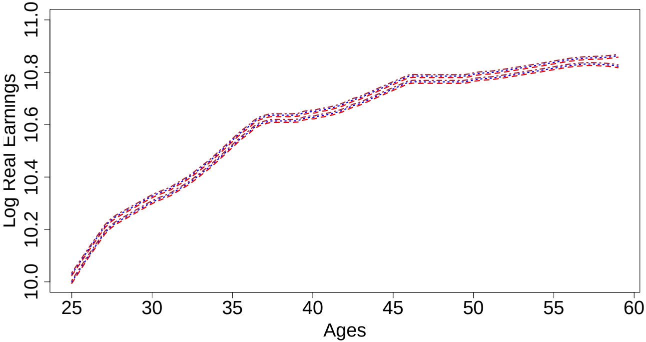


Figure 10: 95% uniform confidence bands for 0.99-quantile smoothers with 3 different education levels "low education", "apprenticeship" & "university"

Real effect of education for income?

- High educations offers a safety line!
- For high end (income) labour, high education no significant effect
    - Smart, no need go to school
    - Be scientist after Ph.D. graduation
    - Poor, not continue school, but hard working & know a lot from practice
    - Education may make people less creative
    - ...
- Causality test, Jeong, Härdle and Song (2009)

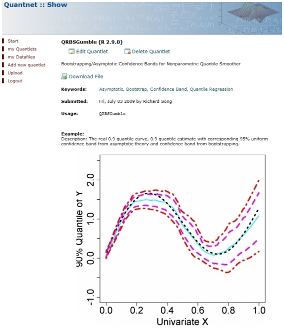
Drawbacks

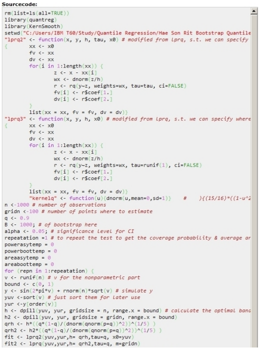
- Very rich people maybe not recorded in the pension system
- Maybe not same retirement time
- Panel data, not exactly i.i.d. (furthur research)
- . . .

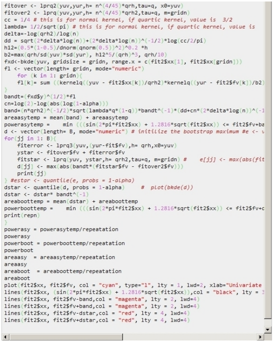
Sth must keep in mind!

- You are dealing with 70-year old people now!
- Time flies (technology level $\uparrow$), more and more high income jobs require high educated people. Time variation of the $\hat{\beta}$? further research.

Quantnet - Open for sharing







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Appendix - Assumptions

λ_i and C_i: generic constants.

A1. $X_1, \ldots, X_n$ are an i.i.d. sample, and $f_X(x) \geq \lambda_0$. The quantile function satisfies: $|l'(\cdot)| \leq \lambda_1,\ |l''(\cdot)| \leq \lambda_2$.

A2. $F(t|x)$ have a density, $f(t|x) \geq \lambda_3 > 0$, continuous in x, and in t in the neighborhood of 0. That is, for some $A(\cdot)$ and $f_0(\cdot)$

$$F(t|x') = p + f_0(x)t + A(x)(x' - x) + R(t, x'; x),$$

where $\sup_{t,x,x'} \frac{|R(t,x';x)|}{t^2+|x'-x|^2} < \infty$.

Note that by Assumption A1, $l_h(x)$ is the quantile of a discrete distribution.
This distribution is equivalent to a sample size of $\mathcal{O}_p(nh)$ from a distribution with p-quantile whose biased is $\mathcal{O}_p(h^2)$ relative to the true value.
Let δ_n be the local rate of convergence of the function l_h, essentially $\delta_n = h_n^2 + (nh_n)^{-1/2} = \mathcal{O}(n^{-2/5})$, with $h_n = \mathcal{O}(n^{-1/5})$.

A3. The estimate l_g, satisfies:

$$\begin{aligned} \sup_{x \in J^*} |l_g''(x) - l''(x)| &= \mathcal{O}_p(1), \\ \sup_{x \in J^*} |l_g'(x) - l'(x)| &= \mathcal{O}_p(\delta_n / h) \end{aligned} \tag{7}$$

Note that there is no S_n term in (7) exactly because the bandwidth g_n used to calculate l_g is slightly larger than that used for l_h. As a result l_g has a slightly worse rate of convergence (as an estimator of the quantile function), but its derivatives converge faster.

We assume:

(A4). $f_X(x)$ is twice continuously differentiable and $f(t|x)$ is uniformly bounded in x and t by, say, λ_4.

(A7). The conditional densities $f(\cdot|y),\ y \in \mathbb{R}$, are uniformly local Lipschitz continuous of order $\tilde{\alpha}$ (ulL-$\tilde{\alpha}$) on J, uniformly in $y \in \mathbb{R}$, with $0 < \tilde{\alpha} \leqslant 1$, and $(nh)/\log n \to \infty$.

(A8). $\inf_{v \in J^*} \left| \int \psi\{y - l(v) + \varepsilon\} dF(y|v) \right| \geqslant \tilde{q}|\varepsilon|, \quad \text{for } |\varepsilon| \leqslant \delta_1,$

where δ_1 and $\tilde{q}$ are some positive constants, see also [?]. This assumption is satisfied if there exists a constant $\tilde{q}$ such that $f\{l(v)|v\} > \tilde{q}/p$, $x \in J$.