Inhomogeneous Dependence Modelling with Time Varying Copulae

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Linear Portfolio

- *value of portfolio* $w = (w_1, \ldots, w_d)^\top$ of assets $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$:

$$V_t = \sum_{j=1}^{d} w_j S_{j,t}$$

- *profit and loss* (P&L) function:

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1)$$

$$X_{t+1} = (\log S_{t+1} - \log S_t)$$

- *Value-at-Risk* at level α:

$$VaR(\alpha) = F_L^{-1}(\alpha)$$

Log returns DCX & VW

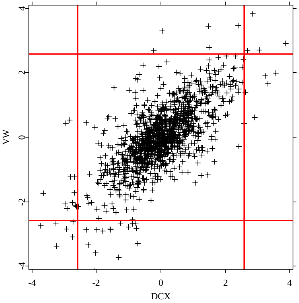


Figure 1: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), quantiles $\Phi^{-1}(0.005)$ and $\Phi^{-1}(0.995)$ (red). maxmindep.xpl

Log returns DCX & VW at 20030408

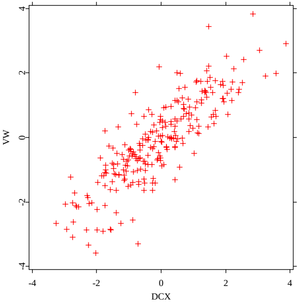


Figure 2: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20020415-20030408. maxmindep.xpl

Log returns DCX & VW at 20041027

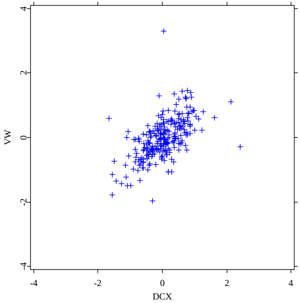


Figure 3: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20031103-20041027. maxmindep.xpl

The VaR depends on the distribution F_X of the risk factor increments $X = (X_1, \dots, X_d)^\top$.

1. How to model the dependency among $X_1, \dots, X_d$?
2. How does F_X and the dependency among $X_1, \dots, X_d$ vary over time ?

Traditional approach

- the conditional distribution of log-returns is multivariate normal: $X_t \sim N(0, \Sigma_t)$
- the covariance matrix Σ_t is estimated by:

$$\widehat{\Sigma}_t = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X_{t-s} X_{t-s}^T$$

- *decay factor* λ $(0 < \lambda < 1)$ is determined by backtesting
- $\lambda = 0.94$ provides best results (Morgan/Reuters, 1996)
- Drawbacks:
  - does not allow to generate tail dependence
  - does not allow heavy tails

Copula based approach

- the conditional distribution of log-returns is modelled with Copula C:

$$X_t \sim C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d), \theta_t\}$$

- $F_{X_1}, \dots, F_{X_d}$ are marginal distributions
- θ_t dependence parameter

(Embrechts, 1999)

A single global copula parameter $\theta_t = \theta$ is too optimistic.
Copula parameter θ_t almost constant on certain intervals.

- find the largest interval *I* for which parametric assumption is acceptable ($\o\rho\alpha\kappa\lambda\epsilon$)
- find this interval for each *t* adaptively
- practically speaking: estimate the dependence parameter θ_t in a time varying interval

Modelling Dependence over Time

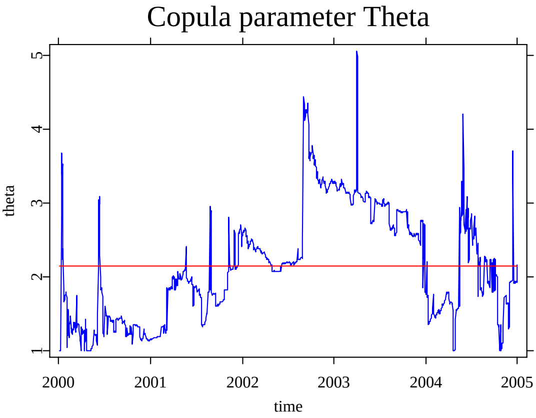


Figure 4: Dependence over time for DaimlerChrysler(DCX) and Volkswagen(VW), 20000103-20041230. plotrealtheta.xpl

Outline

Copulae

Theorem (Sklar's theorem)

For a distribution function F with marginals $F_{X_1}, \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$ with

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \tag{1}$$

If $F_{X_1}, \ldots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \ldots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \ldots, F_{X_d}$.

With copula density

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}$$

the density function of $F(x_1, \ldots, x_d)$ is

$$f(x_1, \ldots, x_d) = c\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \prod_{j=1}^{d} f_j(x_j)$$

where $u_j = F_{X_j}(x_j)$ and $f_j(x_j) = F'_{X_j}(x_j)$, $j = 1 \ldots d$

1. Gaussian Copula

$$C_{\Psi}^{Ga}(u_1, \dots, u_d) = \Phi_{\Psi}\{\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\}$$

Φ univariate standard normal cdf

Φ_{Ψ} d-dimensional standard normal cdf with correlation matrix Ψ

- Gaussian copula contains *the dependence structure*
- *normal* marginal distributions + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distributions + Gaussian copula = *meta-Gaussian* distributions

Explicit expression for the Gaussian copula

$$
\begin{aligned}
C^{Ga}_{\Psi}(u_1,\dots,u_d) &= \Phi_{\Psi}\{\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_d)\} \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} e^{(-\frac{1}{2}r^{\top}\Psi^{-1}r)} dr_1 \dots dr_d
\end{aligned}
$$

where

$r = (r_1,\dots,r_d)^{\top}$, $u_j = \Phi(x_j)$

- $C^{Ga}_{\Psi}(u_1,\dots,u_d)$ allows to generate joint symmetric dependence, but no tail dependence (i.e., there are no joint extreme events)

2. Frank Copula, $0 < \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = -\frac{1}{\theta} \log \left[1 + \frac{\prod_{j=1}^{d} \left\{ \exp(-\theta u_j) - 1 \right\}}{\{\exp(-\theta) - 1\}^{d-1}} \right]$$

- dependence becomes maximal when $\theta \longrightarrow \infty$
- independence is achieved when $\theta = 0$

3. Gumbel-Hougaard copula, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = \exp\left[- \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{\theta^{-1}} \right]$$

- for $\theta > 1$ allows to generate dependence in the upper tail (Schmidt, 2005)
- For $\theta = 1$ reduces to the product copula, i.e. $C_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$.
- for $\theta \longrightarrow \infty$, we obtain the Fréchet-Hoeffding upper bound:

$$C_\theta(u_1, \ldots, u_d) \stackrel{\theta \to \infty}{\longrightarrow} \min(u_1, , \ldots, u_d).$$

4. Ali-Mikhail-Haq copula, $-1 \leq \theta < 1$

$$C_\theta(u_1, \ldots, u_d) = \frac{\prod_{j=1}^{d} u_j}{1 - \theta \left\{ \prod_{j=1}^{d} \left(1 - u_j\right) \right\}}$$

- independence is achieved when $\theta = 0$
- the Fréchet-Hoeffding bounds are not achieved

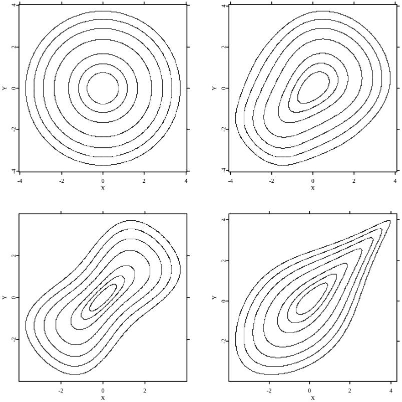


Figure 5: Pdf contour plots, $F(x_1, x_2) = C\{\Phi(x_1), \Phi(x_2)\}$ with Gaussian ($\rho = 0$), AMH ($\theta = 0.9$), Frank ($\theta = 8$), Gumbel ($\theta = 2$) copulae. cont4.xpl

5. Clayton copula, $\theta > 0$

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left(\sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-\theta^{-1}}$$

- dependence becomes maximal when $\theta \longrightarrow \infty$
- independence is achieved when $\theta \longrightarrow 0$
- the distribution tends to the lower Fréchet-Hoeffding bound when $\theta \longrightarrow 1$
- allows to generate asymmetric dependence and lower tail dependence, but no upper tail dependence

Value-at-Risk with Copulae

The process $\{X_t\}_{t=1}^{T}$ of log-returns can be modelled as

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t}$$

with $E[\varepsilon_{j,t}] = 0$, $E[\varepsilon_{j,t}^2] = 1, j = 1, \dots, d$ and

$$E[X_{j,t} \mid \mathcal{F}_{t-1}] = \mu_{j,t}$$

$$E[(X_{j,t} - \mu_{j,t})^2 \mid \mathcal{F}_{t-1}] = \sigma_{j,t}^2$$

where $\mathcal{F}_t$ is the available information at time t.

- $\varepsilon_t = (\varepsilon_{1,t}, \dots, \varepsilon_{d,t})^\top$ are standardised *i.i.d.* innovations with a joint distribution function F_ε
- ε_j, $j = 1, \dots, d$ have continuous marginal distributions F_j

VaR with Copulae

For the log-returns $\{x_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$ Value-at-Risk at level α is estimated:

1. determination of the innovations $\hat{\varepsilon}_t$ (e.g. by deGARCHing)
2. specification and estimation of marginal distributions $F_j(\hat{\varepsilon}_j)$
3. specification of a copula C and estimation of dependence parameter θ
4. simulation of innovations ε and losses L
5. determination of $\widehat{VaR}(\alpha)$, the empirical α-quantile of F_L.

Copula estimation

The distribution of $X = (X_1, ..., X_d)^\top$ with marginals $F_{X_j}(x_j, \delta_j)$, $j = 1, ..., d$ is given by:

$$F_X(x_1, \ldots, x_d) = C\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\}$$

and its density is given by

$$f(x_1, \ldots, x_d; \delta_1, \ldots, \delta_d, \theta)$$

$$= c\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j)$$

where c is a copula density.

For a sample of observations $\{x_t\}_{t=1}^{T}$ and $\vartheta = (\delta_1, \ldots, \delta_d, \theta)^\top \in \mathbb{R}^{d+1}$ the likelihood function is

$$L(\vartheta; x_1, \ldots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \ldots, x_{d,t}; \delta_1, \ldots, \delta_d, \theta)$$

and the corresponding log-likelihood function

$$\ell(\vartheta; x_1, \ldots, x_T) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \delta_1), \ldots, F_{X_d}(x_{d,t}; \delta_d); \theta\}$$

$$+ \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{j,t}; \delta_j)$$

Full Maximum Likelihood (FML)

- FML estimates vector of parameters ϑ in one step through

$$\tilde{\vartheta}_{FML} = \arg\max_{\vartheta} \ell(\vartheta).$$

- the estimates $\tilde{\vartheta}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve

$$(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

- Drawback: with an increasing dimension the algorithm becomes too burdensome computationally.

Inference for Margins (IFM)

1. estimate parameters δ_j from the marginal distributions:

$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j) = \arg\max_{\delta} \left\{ \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j) \right\}$$

2. estimate the dependence parameter θ by maximizing the *pseudo log-likelihood* function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}$$

- The estimates $\hat{\vartheta}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

- Advantage: numerically stable.

Moving Window

- use static windows of size $w = 250$ scrolling in time t for VaR estimation:

  $$\{x_t\}_{t=s-w+1}^{s}$$

  for $s = w, \ldots, T$

- the VaR estimation procedure generates a time series $\{\widehat{VaR}_t\}_{t=w}^{T}$ and $\{\hat{\theta}_t\}_{t=w}^{T}$ of dependence parameters estimates.

Adaptive Copula estimation

Using *Local Change Point detection (LoChaP)*(Mercurio, Spokoiny, 2004) we sequentially test: θ_t is constant (i.e. $\theta_t = \theta$) within some interval I (local parametric assumption).
”Oracle“ choice: the largest interval $I = [\nu, n[$, for which the *small modelling bias condition (SMB)*

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta$$

is fulfilled.

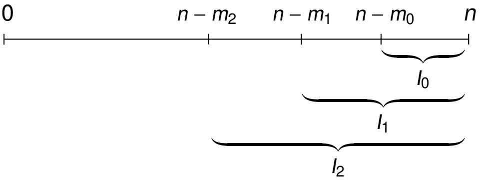
- ν denotes the change point
- θ is then estimated from the interval $I = [\nu, n[$

Choice of the interval of homogeneity

LoChaP is based on adaptive choice of the interval of homogeneity for the endpoint n.

Define $\mathcal{I} = \{I_k, k = 0, 1, ...\}$ such that $I_k = [n - m_k, n]$ with m_k:
$m_0 < m_1 < m_2 < ... \leq n$

$$H_0\text{: copula parameter } \theta_t \text{ is constant within interval } I_k$$



LoChaP procedure

1. start from the smallest interval I_0
2. test the H_0 hypothesis of homogeneity within I_0
3. if H_0 not rejected, take the next larger interval
4. continue the procedure until a possible change point $\hat{\nu}$ is detected or the largest possible interval $[0, n[$ is reached
5. if H_0 is within some I_k rejected, the estimated interval of homogeneity $\widehat{I} = [\hat{\nu}, n[$ or $\widehat{I} = [0, n[$
6. estimate copula dependence parameter θ from observation S_t for $t \in \widehat{I}$, assuming the homogeneous model within $\widehat{I}$, i.e., define $\hat{\theta}_n = \hat{\theta}_{\widehat{I}}$

Test of homogeneity against a change point alternative

Let $I = [n - m, n[$ be an interval candidate.
Let $\mathcal{T}_I$ be a set of internal points within I

H_0: $\theta_t = \theta \; \forall \tau \in \mathcal{T}_I$
H_1: $\exists \tau \in \mathcal{T}_I$: $\theta_t = \theta_1$ for $t \in J = [\tau, n[$ and $\theta_t = \theta_2$ for $t \in J^c = I - J = [n - m, \tau[$

- log-likelihood $\ell_I(\theta)$ corresponding to H_0
- log-likelihood $\ell_J(\theta_1) + \ell_{J^c}(\theta_2)$ corresponding to H_1

Test of homogeneity against a change point alternative

Likelihood ratio test for the fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2} \{\ell_J(\theta_1) + \ell_{J^c}(\theta_2)\} - \max_{\theta} \ell_I(\theta) \\ &= \ell_J(\hat{\theta}_J) + \ell_{J^c}(\hat{\theta}_{J^c}) - \ell_I(\hat{\theta}_I) \\ &= \hat{\ell}_J + \hat{\ell}_{J^c} - \hat{\ell}_I \end{aligned}$$

Test statistics for unknown change point location:

$$T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}$$

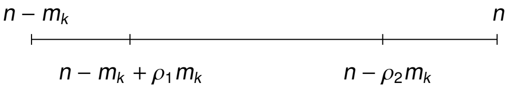
Reject H_0 if $T_I > \lambda_I$

Change point $\hat{\nu} = \arg\max_{\tau \in \mathcal{T}_{\mathcal{I}}} T_{I,\tau}$

Implementation Example

Selection of interval candidates $\mathcal{I}$: $\mathcal{I} = \{I_k : I_k = [n - m_k, n[\}$ with $m_k = \left[m_0 c^k\right]$, $k = 0, 1, 2, ...$ for some $c > 1$.

Setting of $\mathcal{T}_{I_k}$: $\mathcal{T}_I = \{t : n - m_k + \rho_1 m_k \leq t \leq n - \rho_2 m_k\}$ for fixed parameters $\rho_1 \leq 1/3$ and $\rho_2 \leq 1/3$.



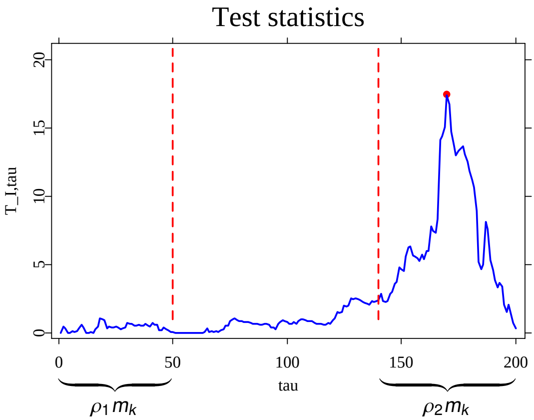


Figure 6: Test statistics $T_{I,\tau}$ for one fixed interval I_k of length $m_k = 200$ plotted against τ. Parameters $\rho_2 = 1/3$ and $\rho_1 = 0.25$ (dotted lines).

testChangePointMod.xpl

Choice of the critical values

Choose critical values λ_I to provide a prescribed first kind error probability α:

1. Set β_I for every I: $\sum_{I \in \mathcal{I}} \beta_I = \alpha$, e.g. for M_{I_k} denoting the number of points in interval I_k we have

$$\beta_{I_k} = \alpha M_{I_k}^{-1} \left(\sum_{I_j \in \mathcal{I}} M_{I_j}^{-1} \right)^{-1} \approx \frac{\alpha(1 - c^{-1})}{c^k}$$

   and the corresponding value α_{I_k}:

$$\alpha_{I_k} \approx \alpha(1 - c^{-(k+1)})$$

2. Select by Monte Carlo critical values λ_{I_k} for every interval I_k:

$$P_{H_0}\left(max_{k' \leq k} T_{I_{k'}} > \lambda_{I_{k'}} \right) = \alpha_{I_k}$$

Simulated Examples

A set of 240 observations was simulated from a bivariate Gumbel-Hougaard copula

$$C_\theta(u, v) = \exp\left[-\left\{(-\log u)^\theta + (-\log v)^\theta\right\}^{1/\theta}\right]$$

with parameter:

$$\theta_t = \begin{cases} 1 & \text{if} \quad 1 \leq t \leq 80 \\ 3 \quad \text{or} \quad 2 & \text{if} \quad 81 \leq t \leq 160 \\ 1 & \text{if} \quad 161 \leq t \leq 240 \end{cases}$$

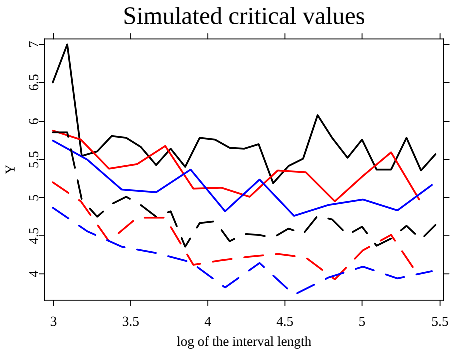


Figure 7: Critical values for $\alpha = 0.05$ (solid line) and $\alpha = 0.1$ (dashed line), computed by simulations using Gumbel-Hougaard copula with parameters $m_0 = 20$, $\rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (black line), $c = 1.2$, $\rho_1 = 0.2$ (red line), $c = 1.25$, $\rho_1 = 0.25$ (blue line).
critplot.xpl

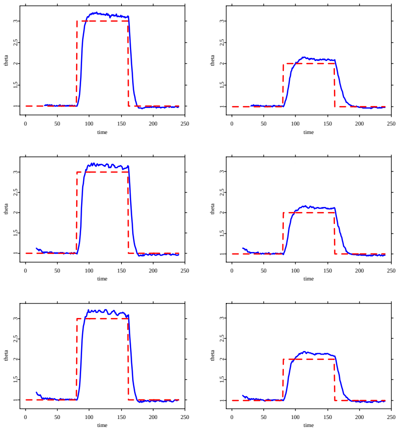


Figure 8: Pointwise mean (blue) based on 200 simulations of the data, simulated from the Gumbel-Hougaard copula and real parameter (dashed); jump size 2 (left panel), jump size equal 1 (right panel); with parameters $m_0 = 20$, $\rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (upper panel); $c = 1.2$, $\rho_1 = 0.2$ (middle panel); $c = 1.25$, $\rho_1 = 0.25$ (lower panel).
thetaplotmean.xpl

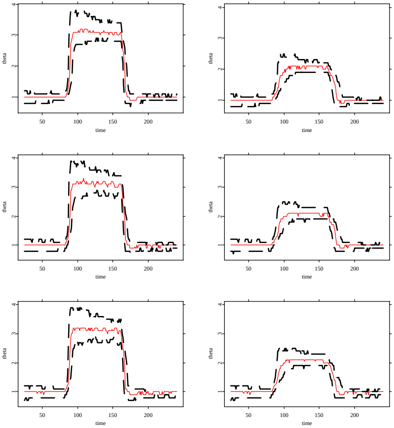


Figure 9: Pointwise median (red) and quartiles (dashed) , based on 200 simulations of the data, simulated from the Gumbel-Hougaard copula; jump size 2 (left panel), jump size equal 1 (right panel); with parameters $m_0 = 20$, $\rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (upper panel); $c = 1.2$, $\rho_1 = 0.2$ (middle panel); $c = 1.25$, $\rho_1 = 0.25$ (lower panel). thetaplot.xpl

Detection delays with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$

to the first jump at $t = 80$	mean	standard deviation	maximum	minimum
40% rule	9.215	3.596	21	1
50% rule	9.475	3.697	21	2
60% rule	9.740	3.860	24	2
to the second jump at $t = 160$	**mean**	**standard deviation**	**maximum**	**minimum**
40% rule	6.175	3.158	15	1
50% rule	6.890	3.216	17	1
60% rule	7.605	3.634	21	2

Table 1: Descriptive statistics for the detection speeds to sudden jumps of the Gumbel-Hougaard copula dependence parameter with a jump size of 2. The results are obtained with parameters $\alpha = 0.05$, $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$ and $\rho_2 = 0.3$. Statistics are based on 200 simulations.

Detection delays with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$

to the first jump at $t = 80$	mean	standard deviation	maximum	minimum
40% rule	10.210	5.829	28	1
50% rule	11.535	5.150	29	1
60% rule	13.030	5.801	37	1
to the second jump at $t = 160$	mean	standard deviation	maximum	minimum
40% rule	8.885	5.428	28	1
50% rule	10.075	5.719	32	1
60% rule	11.800	7.880	61	1

Table 2: Descriptive statistics for the detection speeds to sudden jumps of the Gumbel-Hougaard copula dependence parameter with a jump size of 1. The results are obtained with parameters $\alpha = 0.05$, $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$ and $\rho_2 = 0.3$. Statistics are based on 200 simulations.

Simulated Examples

A set of 240 observations was simulated from a bivariate Clayton copula

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

with parameter:

$$\theta_t = \begin{cases} 1 & \text{if} \quad 1 \le t \le 80 \\ 3 \quad \text{or} \quad 2 & \text{if} \quad 81 \le t \le 160 \\ 1 & \text{if} \quad 161 \le t \le 240 \end{cases}$$

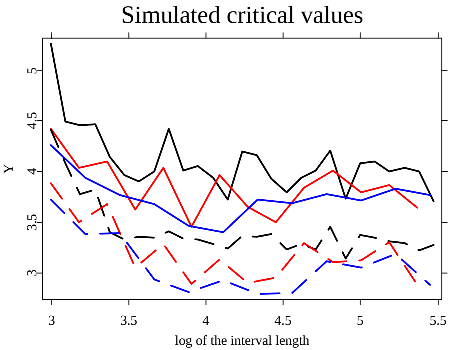


Figure 10: Critical values for $\alpha = 0.05$ (solid line) and $\alpha = 0.1$ (dashed line), computed by simulations using Clayton copula with parameters $m_0 = 20$, $\rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (black line), $c = 1.2$, $\rho_1 = 0.2$ (red line), $c = 1.25$, $\rho_1 = 0.25$ (blue line). critplot.xpl

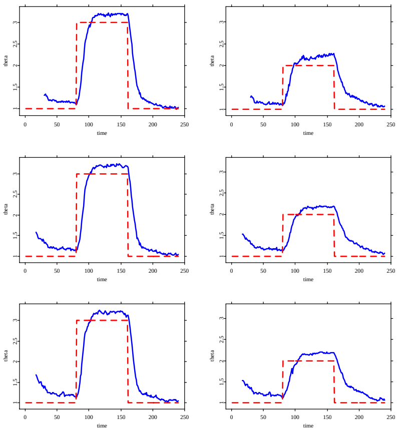


Figure 11: Pointwise mean (blue) based on 200 simulations of the data, simulated from the Clayton copula and real parameter (dashed); jump size 2 (left panel), jump size equal 1 (right panel); with parameters $m_0 = 20, \rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (upper panel); $c = 1.2$, $\rho_1 = 0.2$ (middle panel); $c = 1.25$, $\rho_1 = 0.25$ (lower panel). thetaplotmean.xpl

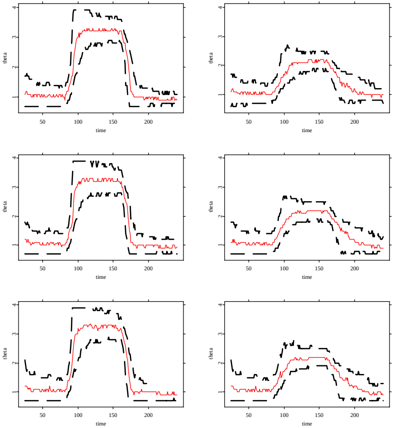


Figure 12: Pointwise median (red) and quartiles (dashed) , based on 200 simulations of the data, simulated from the Clayton copula; jump size 2 (left panel), jump size equal 1 (right panel); with parameters $m_0 = 20$, $\rho_2 = 0.3$ and $c = 1.1$, $\rho_1 = 0.1$ (upper panel); $c = 1.2$, $\rho_1 = 0.2$ (middle panel); $c = 1.25$, $\rho_1 = 0.25$ (lower panel). thetaplot.xpl

Detection delays with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$

to the first jump at $t = 80$	mean	standard deviation	maximum	minimum
40% rule	10.940	6.519	32	1
50% rule	12.440	7.326	39	1
60% rule	13.410	7.708	39	1
to the second jump at $t = 160$	**mean**	**standard deviation**	**maximum**	**minimum**
40% rule	9.688	6.716	36	1
50% rule	10.764	7.416	53	1
60% rule	12.352	8.109	53	1

Table 3: Descriptive statistics for the detection speeds to sudden jumps of the Clayton copula dependence parameter with a jump size of 2. The results are obtained with parameters $\alpha = 0.05$, $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$ and $\rho_2 = 0.3$. Statistics are based on 200 simulations.

Detection delays with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$

to the first jump at $t = 80$	mean	standard deviation	maximum	minimum
40% rule	10.100	10.051	60	1
50% rule	11.745	10.918	60	1
60% rule	13.870	11.912	60	1
to the second jump at $t = 160$	mean	standard deviation	maximum	minimum
40% rule	16.626	14.837	74	1
50% rule	19.843	17.547	75	1
60% rule	21.727	18.064	79	1

Table 4: Descriptive statistics for the detection speeds to sudden jumps of the Clayton copula dependence parameter with a jump size of 1. The results are obtained with parameters $\alpha = 0.05$, $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$ and $\rho_2 = 0.3$. Statistics are based on 200 simulations.

Kullback-Leibler Divergence

- *Kullback-Leibler divergence* is defined as:

$$\mathcal{K}(P_\vartheta, P_{\vartheta'}) = E_\vartheta \log \frac{p(y,\vartheta)}{p(y,\vartheta')}$$

	$K(1,2)$	$K(1,3)$	$K(2,1)$	$K(3,1)$
Gumbel-Hougaard	123.01	339.36	79.377	158.08
Clayton	24.938	84.439	18.096	48.803

Table 5: Kullback-Leibler information number $K(\theta_1, \theta_2)$ and $K(\theta_2, \theta_1)$ for fixed $\theta_1 = 1$ and parameter $\theta_2 = 2.0, 3.0$; for the Gumbel-Hougaard and the Clayton copula.

- density $p(y,\vartheta) = dP_\vartheta(y)/dP$
- P_ϑ is dominated by a σ-finit measure P
- P_ϑ belongs to some parametric family $\mathbb{P}$

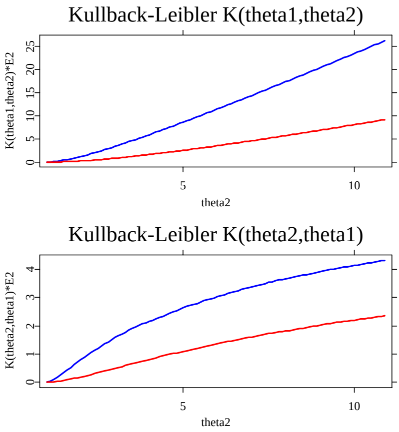


Figure 13: Kullback-Leibler information number $K(\theta_1, \theta_2)$ (upper panel) and $K(\theta_2, \theta_1)$ (lower panel) plotted against θ_2 for fixed $\theta_1 = 1$. The blue line refers to the Gumbel-Hougaard copula and the red line to the Clayton copula. KullbackLeibler.xpl

Applications

Data sets from the DAX portfolio: DaimlerChrysler, Volkswagen, Bayer, BASF, Allianz and Münchener Rückversicherung, 20000101-20031231.

Marginal parameters $\hat{\delta}_{t,j} = \hat{\sigma}^2_{t,j}$, $j = 1, 2$ (DaimlerChrysler, Volkswagen) are estimated at time t by exponential smoothing:

$$\hat{\delta}_{t,j} = \hat{\sigma}^2_{t,j} = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X^2_{s,j}$$

$X_{s,j}$ denotes log returns at time s and $0 \leq \lambda \leq 1$ is a smoothing parameter (set $\lambda = 1/20$).

Choose Gumbel-Hougaard copula:

$$C_{\theta}(u, v) = \exp\left[-\left\{(-\log u)^{\theta} + (-\log v)^{\theta}\right\}^{1/\theta}\right]$$

Recall: $\theta = 1$ indicates independence.

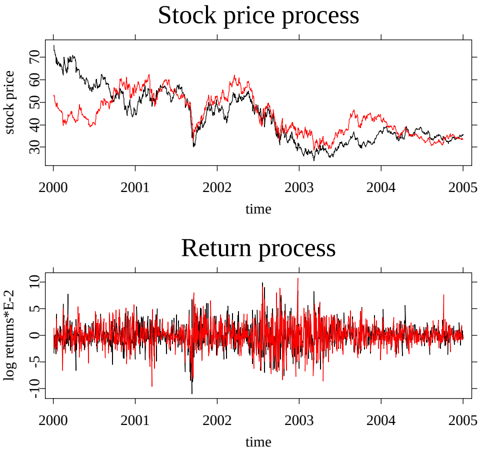


Figure 14: Stock price process (upper panel), log returns (middle panel) and copula dependence parameter θ (lower panel) for DaimlerChrysler (black line) and Volkswagen (red line). The estimates of θ are obtained with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$ and $\alpha = 0.05$. plotDCXVW.xpl

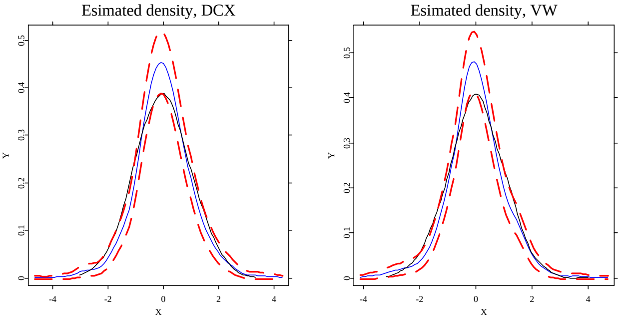


Figure 15: Kernel density estimator of the residuals from DaimlerChrysler (left panel, blue line) and Volkswagen (right panel, blue line) and of the normal density (black line); confidence bands (dashed red lines) at level 0.05. Quartic Kernel is used with $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.

densest.xpl

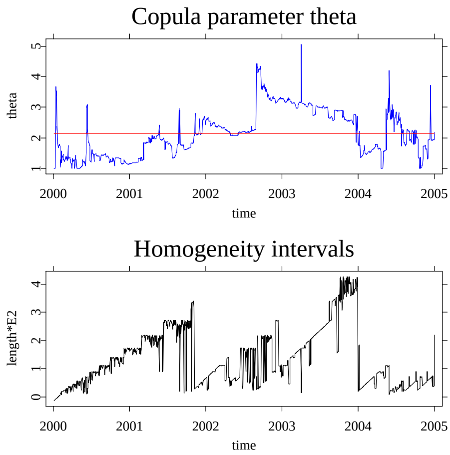


Figure 16: Upper panel: estimated copula dependence parameter θ for DaimlerChrysler and Volkswagen (blue line) and its mean (red line). Lower panel: estimated intervals of time homogeneity. The results are obtained with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$ and $\alpha = 0.05$. realthetahomlength.xpl

Applications

Marginal parameters $\hat{\delta}_{t,j} = \hat{\sigma}^2_{t,j}$, $j = 1, .., 6$ are estimated at time t by exponential smoothing:

$$\hat{\delta}_{t,j} = \hat{\sigma}^2_{t,j} = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X^2_{s,j}$$

Choose Gumbel-Hougaard copula since it allows to generate a lower tail dependence that is crucial for VaR estimation:

$$C_{\theta}(u_1, ..., u_d) = (u_1^{-\theta} + ... + u_d^{-\theta} - d + 1)^{-1/\theta}$$

with copula density

$$c_{\theta}(u_1, ..., u_d) = \prod_{j=1}^{d} \{1 + (j-1)\theta\} \prod_{j=1}^{d} u_j^{-(\theta+1)} \left\{ \sum_{j=1}^{d} u_j^{-\theta} - d + 1 \right\}^{-(1/\theta + d)}$$

Recall: $\theta = 0$ indicates independence.

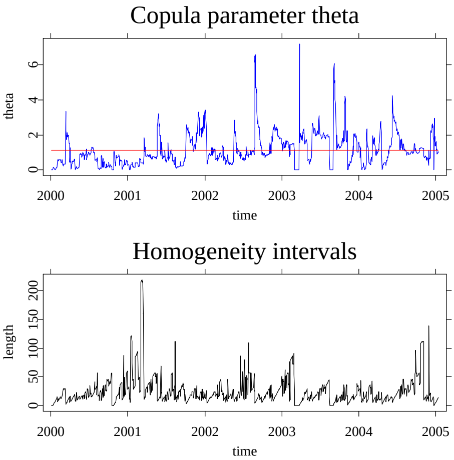


Figure 17: Upper panel: estimated copula dependence parameter θ for 4-dim data: DaimlerChrysler, Volk-swagen, Bayer and BASF (blue) and its mean (red). Lower panel: estimated intervals of time homogeneity; with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$ and $\alpha = 0.05$. realthetahomlength.xpl

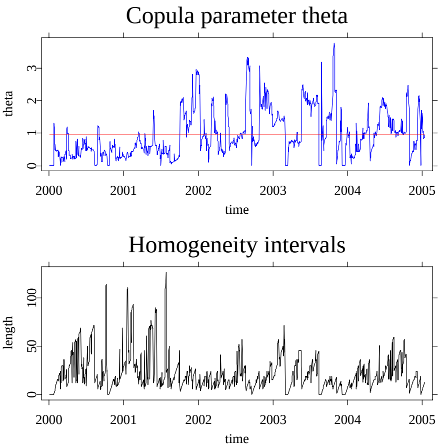


Figure 18: Upper panel: estimated copula dependence parameter θ for 6-dim data: DaimlerChrysler, Volk-swagen, Bayer, BASF, Allianz and Münchener Rückversicherung (blue) and its mean (red). Lower panel: estimated intervals of time homogeneity; with parameters $m_0 = 20$, $c = 1.25$, $\rho_1 = 0.25$, $\rho_2 = 0.3$ and $\alpha = 0.05$. realthetahomlength.xpl

Backtesting

compare the estimated values for the VaR with the true realizations $\{l_t\}$ of the P&L function

the *exceedances ratio* is given by

$$\hat{\alpha} = \frac{1}{T-w} \sum_{t=w}^{T} \mathbf{1}\{l_t < \widehat{VaR}_t(\alpha)\}$$

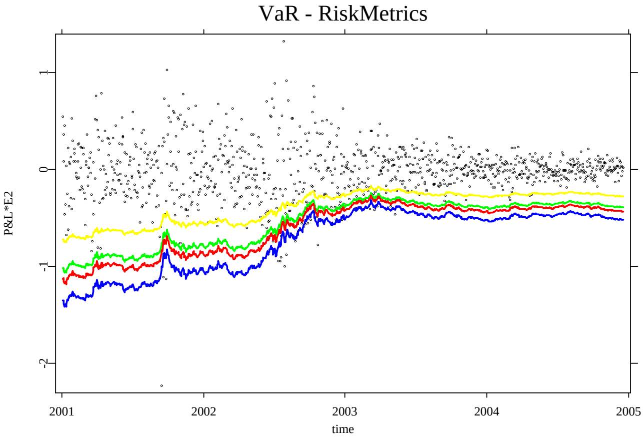
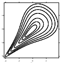


Figure 19: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.05$, $\alpha_3 = 0.005$, $\alpha_1 = 0.005$, $\alpha_1 = 0.001$, $w = (3,2,3,2,3,-1)^\top$, estimated using RiskMetrics approach. RiskMetrics.xpl

Table 6: Exceedances ratio $\hat{\alpha}$ for different portfolios, calculated using Riskmetrics.

	Exceedances ratio $\alpha(\times 10^2)$			
Portfolio	5	1	0.5	0.1
$(1,1,1,1,1,1)$	7.4583	3.6310	3.0422	1.7664
$(1,2,3,2,1,3)$	7.5564	3.631	3.0422	1.6683
$(2,1,2,3,1,3)$	6.6732	3.4347	2.6497	1.5702
$(3,2,3,2,3,1)$	9.2247	4.1217	3.7291	2.3553
$(3,1,2,1,3,2)$	8.4396	3.7291	3.3366	2.1590
$(1,3,1,2,3,1)$	5.3974	2.9441	2.3553	0.68695
$(2,1,3,2,1,3)$	8.2434	3.8273	3.6310	2.0608
$(2,3,3,2,1,1)$	7.8508	3.8273	3.3366	2.1590
$(3,1,2,2,2,3)$	7.8508	3.7291	3.1403	1.8646
....	......	......	......	......
$(3,2,3,2,3,-1)$	10.3040	4.8086	4.0236	2.6497
$(3,1,2,1,3,-2)$	12.3650	7.7527	5.6919	4.1217
$(1,3,1,2,3,-1)$	6.3788	3.5329	2.5515	1.1776
$(2,1,3,2,1,-3)$	7.8508	3.7291	3.2385	1.6683
$(2,3,3,2,1,-1)$	8.6359	4.2198	3.6310	2.4534
$(3,1,2,2,2,-3)$	8.7341	4.2198	3.4347	2.2571
$(2,3,1,1,2,-3)$	10.2060	5.0049	4.4161	2.7478
$(2,3,2,3,2,-3)$	6.7713	3.3366	2.8459	1.1776
$(3,2,3,2,3,-3)$	10.0100	4.9068	4.1217	2.9441
avg.	7.8958	3.9091	3.2180	1.8973
std.dev.	1.8337	1.0768	0.8886	0.8178
$\sum_{w\in W}(\hat{\alpha}-\alpha)^2$	2.7859	2.2977	1.9547	0.9291
$\sum_{w\in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.5572	2.2977	3.9093	9.2909



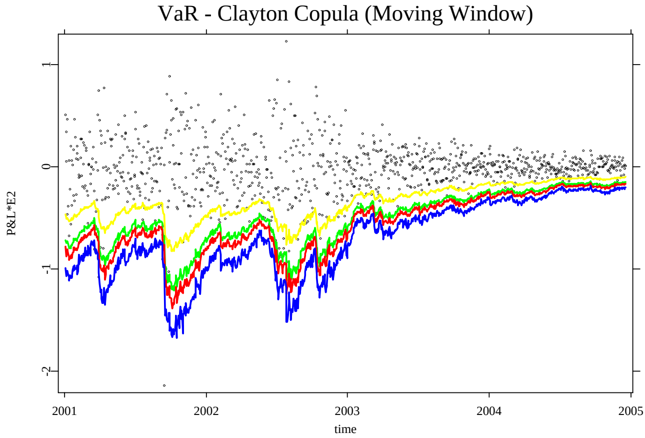


Figure 20: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.05$, $\alpha_3 = 0.005$, $\alpha_1 = 0.005$, $\alpha_1 = 0.001$, $w = (3, 2, 3, 2, 3, -1)^\top$, estimated with Clayton copula using moving window approach. MovingWindow.xpl

Table 7: Exceedances ratio $\hat{\alpha}$ for different portfolios, calculated using Moving Window.

	Exceedances ratio $\alpha(\times 10^2)$			
Portfolio	5	1	0.5	0.1
$(1,1,1,1,1,1)$	6.6732	1.2758	0.7851	0.3925
$(1,2,3,2,1,3)$	6.9676	1.1776	0.5888	0.3925
$(2,1,2,3,1,3)$	6.9676	1.2758	0.5888	0.3925
$(3,2,3,2,3,1)$	6.9676	1.0795	0.7851	0.3925
$(3,1,2,1,3,2)$	6.5751	1.4720	0.6869	0.2944
$(1,3,1,2,3,1)$	6.3788	1.3739	0.6869	0.1963
$(2,1,3,2,1,3)$	7.1639	1.2758	0.5888	0.3925
$(2,3,3,2,1,1)$	7.1639	1.3739	0.6869	0.3925
$(3,1,2,2,2,3)$	6.6732	1.0795	0.7851	0.2944
....	......	......	......	......
$(3,2,3,2,3,-1)$	7.0658	1.1776	0.6869	0.3925
$(3,1,2,1,3,-2)$	6.8695	1.3739	0.6869	0.2944
$(1,3,1,2,3,-1)$	6.5751	1.2758	0.7851	0.2944
$(2,1,3,2,1,-3)$	7.262	1.3739	0.8832	0.4907
$(2,3,3,2,1,-1)$	6.8695	1.472	0.6869	0.3925
$(3,1,2,2,2,-3)$	7.0658	1.3739	0.7851	0.4907
$(2,3,1,1,2,-3)$	6.5751	1.2758	0.8832	0.2944
$(2,3,2,3,2,-3)$	6.9676	1.4720	0.8832	0.3925
$(3,2,3,2,3,-3)$	6.869	1.4720	0.7851	0.3925
avg.	6.8449	1.3371	0.7565	0.3762
std.dev.	0.2854	0.1352	0.0937	0.0899
$\sum_{w \in W}(\hat{\alpha}-\alpha)^2$	0.8356	0.0315	0.0178	0.0202
$\sum_{w \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.1671	0.0315	0.0356	0.2016

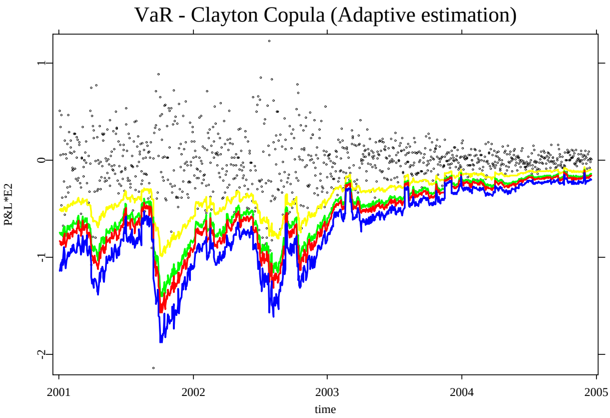


Figure 21: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.05$, $\alpha_3 = 0.005$, $\alpha_1 = 0.005$, $\alpha_1 = 0.001$, $w = (3, 2, 3, 2, 3, -1)^\top$, estimated with Clayton copula using adaptive estimation procedure. LoChaP.xpl

Table 8: Exceedances ratio $\hat{\alpha}$ for different portfolios, calculated using LoChaP procedure.

Portfolio	Exceedances ratio $\alpha(\times 10^2)$			
	5	1	0.5	0.1
$(1, 1, 1, 1, 1, 1)$	7.5564	1.7664	0.9814	0.3925
$(1, 2, 3, 2, 1, 3)$	7.5564	1.7664	0.8832	0.3925
$(2, 1, 2, 3, 1, 3)$	7.1639	1.6683	0.9813	0.3925
$(3, 2, 3, 2, 3, 1)$	7.5564	1.7664	0.9814	0.2944
$(3, 1, 2, 1, 3, 2)$	7.3602	1.9627	1.1776	0.3925
$(1, 3, 1, 2, 3, 1)$	6.8695	1.6683	0.7851	0.1963
$(2, 1, 3, 2, 1, 3)$	7.6546	1.7664	0.8832	0.4907
$(2, 3, 3, 2, 1, 1)$	7.7527	1.6683	0.8832	0.2944
$(3, 1, 2, 2, 2, 3)$	7.5564	1.9627	1.0795	0.2944
....	......	......	......	......
$(3, 2, 3, 2, 3, -1)$	7.5564	1.4720	0.9814	0.2944
$(3, 1, 2, 1, 3, -2)$	7.1639	1.5702	0.88322	0.1963
$(1, 3, 1, 2, 3, -1)$	6.9676	1.668	0.6869	0.1963
$(2, 1, 3, 2, 1, -3)$	7.6546	1.4720	0.8832	0.3925
$(2, 3, 3, 2, 1, -1)$	7.6546	1.6683	0.8832	0.2944
$(3, 1, 2, 2, 2, -3)$	7.6546	1.4720	0.9814	0.2944
$(2, 3, 1, 1, 2, -3)$	6.3788	1.2758	0.6869	0.3925
$(2, 3, 2, 3, 2, -3)$	7.1639	1.5702	0.9813	0.1963
$(3, 2, 3, 2, 3, -3)$	7.3602	1.6683	0.9813	0.2944
avg.	7.3561	1.6519	0.9364	0.3189
std.dev	0.3393	0.1912	0.1417	0.0880
$\sum_{w \in W}(\hat{\alpha} - \alpha)^2$	1.3587	0.1104	0.0503	0.0133
$\sum_{w \in W}(\hat{\alpha} - \alpha)^2/\alpha$	0.2717	0.1104	0.1006	0.1328

Summary

Table 9: Relative squared deviation $\sum_{w \in W} (\hat{\alpha} - \alpha)^2 / \alpha$ for Riskmetrics, Moving Window and LoChaP approach.

	Exceedances ratio $\alpha (\times 10^2)$			
Method	5	1	0.5	0.1
Riskmetrics	0.5572	2.2977	3.9093	9.2909
Moving Window	0.1671	0.0315	0.0356	0.2016
LoChaP	0.2717	0.1104	0.1006	0.1328

Conclusion

- Copula was used to estimate the Value-at-Risk from the 6-dimensional portfolio (DCX, VW, ALV, MUV, BAY and BAS) using Riskmetrics, adaptive estimation and moving window approach
- Backtesting is used to compare the performance of the copula-based Value-at-Risk estimation with a Value-at-Risk estimation using Riskmetrics approach
- All three methods overestimate on average the Value-at-Risk

Conclusion

- The adaptive procedure allows a dynamic selection of the estimation interval for dependence structure
- The moving window and adaptive copula outperform a Riskmetrics in a sense of Value-at-Risk estimation
- The performance of Value-at-Risk for a fixed estimation interval (moving window) is at least as good as with an adaptive method

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Appendix I

For all $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, every copula C satisfies

$$W(u_1, \ldots, u_d) \leq C(u_1, \ldots, u_d) \leq M(u_1, \ldots, u_d)$$

where

$$M(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d)$$

and

$$W(u_1, \ldots, u_d) = \max\left(\sum_{i=1}^{d} u_i - d + 1, 0\right)$$

- $M(u_1, \ldots, u_d)$ is called *Fréchet-Hoeffding upper bound*
- $W(u_1, \ldots, u_d)$ is called *Fréchet-Hoeffding lower bound*

(Hoeffding, 1940)

Appendix II

For a random vector $X = (X_1, X_2)^\top$

- *upper tail dependence coefficient* is defined as

$$\delta = \lim_{u \to 1} P\left\{X_1 > F_{X_1}^{-1}(u) \middle| X_2 > F_{X_2}^{-1}(u)\right\}$$

- *lower tail dependence coefficient* is defined as

$$\gamma = \lim_{u \to 0} P\left\{X_1 \leq F_{X_1}^{-1}(u) \middle| X_2 \leq F_{X_2}^{-1}(u)\right\}$$

X is *upper/lower* tail dependent if $\delta > 0$ resp. $\gamma > 0$.

- *upper tail dependence coefficient for Copula C:*

$$\delta = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u}$$

- *lower tail dependence coefficient for Copula C:*

$$\gamma = \lim_{u \to 0} \frac{C(u, u)}{u}$$

  - Gaussian copula: $\delta = \gamma = 0$
  - Clayton copula: $\delta = 0$, $\gamma = 2^{1/\theta}$
  - Gumbel copula: $\delta = 2 - 2^{1/\theta}$, $\gamma = 0$

Appendix III

- let $\mathbb{P}$ denote the underlying measure, i.e.

$$X_t \sim C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d), \theta_t\} = \mathbb{P}$$

- let $\mathbb{P}_\theta$ denote parametric measure corresponding to the model with a constant parameter θ
- *Small modelling bias condition (SMB)* for an interval I:

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta,$$

  where θ is constant.

  $\mathcal{K}(P_\vartheta, P_{\vartheta'}) = E_\vartheta \log \frac{p(y,\vartheta)}{p(y,\vartheta')}$ denotes *Kullback-Leibler divergence*

Copula dependence parameter θ_t depends on the interval *I*

- “Oracle” (ideal) choice: largest interval *I* such that *SMB* holds
- Aim: mimic “Oracle” choice
- practically speaking: estimate the dependence parameter in time varying interval

Theorem 1 (parametric case):

Under the local parametric assumption θ can be estimated by $\widetilde{\theta}_I$ in an interval I such that it holds:

$$E_{\mathbb{P}_\theta} \left| L\left(\widetilde{\theta}_I, \theta\right) \right|^r \leq \mathcal{R}_{2r}$$

- $\mathcal{R}_{2r}$ is a constant
- L denotes log-likelihood ratio

Theorem 2 (non parametric case):

Under the *SMB* condition (i.e. $\Delta_I(\theta) \leq \Delta$) it holds

$$E_{\mathbb{P}} \left| L\left(\widetilde{\theta}_I, \theta\right) \right|^{r/2} \leq \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta),$$

i.e. $\widetilde{\theta}_I$ is a “good” estimator of θ in an interval I.

- $\mathcal{R}_{2r}$ due to Theorem 1
- “good” means “of parametric quality” up to the factor $\exp(\Delta)$, which is a payment for the model misspecification (approximation of a non parametric by a local parametric)

Theorem 3 (critical values):

There are exist ι_0, ι such that holds $\lambda_{I_k} \leq \iota_0 \log K + \iota(K - k)$.

- simplified procedure: select ι_0, ι due to Theorem 1 such that holds

$$E_{\mathbb{P}_\theta} \left| L\left(\widetilde{\theta}_{I_k}, \hat{\theta}_{I_k}\right) \right|^r \leq \rho \mathcal{R}_{2r}$$

- ρ is a constant
- under the local parametric assumption $\hat{\theta}_{I_k}$ is a“good” estimator of $\widetilde{\theta}_{I_k}$, i.e. $\hat{\theta}_{I_k}$ provides “Oracle” quality

Theorem 4 ("Oracle"):

For $\widetilde{\theta}_I^*$ denoting "Oracle" estimator, $\hat{\theta}$ provides a "good" estimator for $\widetilde{\theta}_I$ in an Interval I, such that under the *SMB* holds:

$$E_{\mathbb{P}} \left| L\left(\widetilde{\theta}_I^*, \hat{\theta}\right) \right|^{r/2} \leq \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta) + \lambda_I.$$

- $\exp(\Delta)$ is a payment for approximation
- λ_I is a payment for adaptation