Dynamic Factor Models in Risk Behaviour

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Pricing Kernels & Risk Aversion

Pricing Kernel (PK) at time t and maturity $\tau = T - t$

$$M_{t,\tau}(S_T) = \frac{u'(S_T)}{u'(S_t)}$$

1. S_t is value at time t from wealth, consumption, asset
2. $u(x)$ is utility function
3. under risk aversion: $M(x)$ monotone decreasing

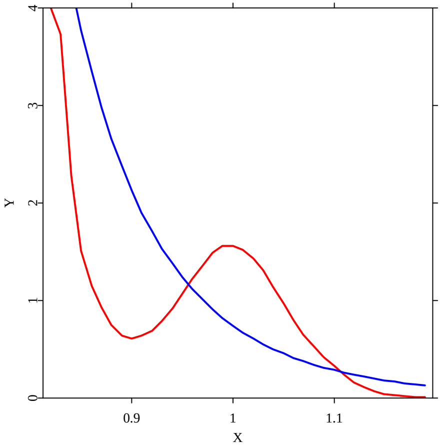


Figure 1: Empirical (red) and theoretical (blue) pricing kernel, DAX 19990205, $\tau = 10$ days.

CARA / CRRA Utility Functions

CARA utility

$$u(x) = -\frac{1}{\alpha} e^{-\alpha x}$$

$\alpha > 0$ is the absolute risk aversion coefficient.

CRRA utility (power utility)

$$u(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma}$$

$\gamma \in (0, 1)$ is the relative risk aversion coefficient.

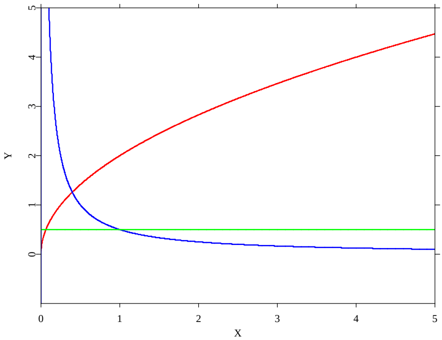


Figure 2: CRRA, $u(x) = \frac{x^\gamma}{\gamma}$ (red), $\alpha(x) = \frac{1-\gamma}{x}$ (blue), $\rho(x) = \gamma$ (green), $\gamma = 0.5$.

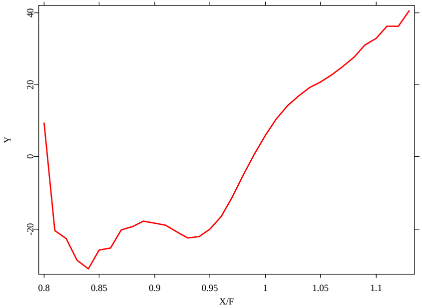


Figure 3: Empirical RRA, DAX 19990205, $\tau = 10$ days.

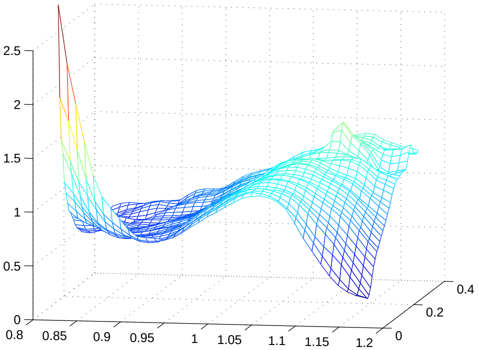


Figure 4: Empirical PK across moneyness κ and maturities τ, DAX 19990303

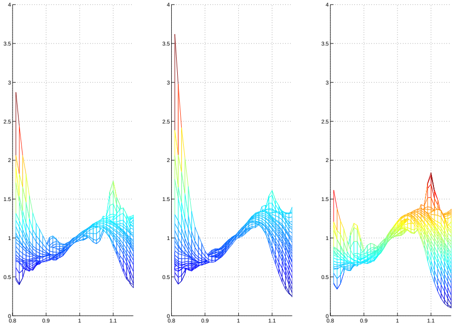


Figure 5: Empirical PK across κ and τ, DAX 19990303, 19990313 and 19990323

Empirical pricing kernels

1. do not reflect risk aversion across all strikes
2. vary across time to maturity τ and time t

$$M(x) = M_{t,\tau}(x)$$

How to explain pricing kernel and risk aversion dynamics ?

Outline

1. Motivation ✓
2. Pricing Kernels
3. DSFM and Pricing Kernel Estimation
4. Empirical Results
5. References

Pricing Kernels

Asset price follows

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu(S_t, t)dt + \sigma(S_t, t)dB_t \\ S_t &= S_0 \exp\left[\left\{\mu(S_t, t) - \frac{1}{2}\sigma^2(S_t, t)\right\} t + \sigma(S_t, t)B_t\right] \end{aligned}$$

where $t \in [0, T]$ and

1. B_t is standard Brownian motion under measure P
2. $B_t^* = B_t + \int_0^t \frac{\mu_s - r}{\sigma_s} ds$ is Brownian motion under measure Q

Measure Q defined by $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \zeta_t$,

$$\zeta_t = \exp\left(-\int_0^t \lambda_u dB_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right)$$

$$\lambda_t = \frac{\mu(S_t, t) - r}{\sigma(S_t, t)}$$

λ is called market price of risk

For any measurable function $\Psi(S_t)$ and $0 \leq s \leq t \leq T$

$$E^Q[\Psi(S_t)|\mathcal{F}_s] = E^P\left[\Psi(S_t)\frac{\zeta_t}{\zeta_s}\middle|\mathcal{F}_s\right]$$

The arbitrage free price at time s for payoff $e^{-r\tau}\psi(S_t)$ is

$$e^{-r\tau}E^Q[\psi(S_t)|\mathcal{F}_s] = E^P\left[\psi(S_t)e^{-r\tau}\frac{\zeta_t}{\zeta_s}\middle|\mathcal{F}_s\right]$$

where $\tau = t - s$

The pricing kernel, or stochastic discount factor is defined as

$$
\begin{aligned}
M_{s,\tau} &= e^{-r\tau}\frac{\zeta_t}{\zeta_s} \\
&= \exp\left(-\int_s^t \lambda_u dB_u - \frac{1}{2}\int_s^t \lambda_u^2 du\right)
\end{aligned}
$$

Example: $\mu, \sigma, \lambda \in \mathbb{R}$ (BS)

$$\begin{aligned}
\frac{\zeta_T}{\zeta_t} &= \exp\left\{-\lambda(B_T - B_t) - \frac{\lambda^2\tau}{2}\right\} \\
&= \exp\left[-\frac{\lambda}{\sigma}\left\{\sigma(B_T - B_t) - \sigma\left(\lambda - \frac{\sigma}{2}\right)\tau\right\} - \frac{\lambda\tau}{2}(\lambda - \sigma)\right] \\
&= \left(\frac{S_T}{S_t}\right)^{-\frac{\lambda}{\sigma}} \exp\left\{\frac{\lambda\tau}{2}(\lambda - \sigma) + r\tau\frac{\lambda}{\sigma}\right\} \\
&= \exp\left[\frac{\left\{\log\left(\frac{S_T}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}^2 - \left\{\log\left(\frac{S_T}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau\right\}^2}{2\sigma^2\tau}\right] \\
&= \frac{q_t(S_T)}{p_t(S_T)}
\end{aligned}$$

Thus, $M_{t,\tau}$ is decreasing in S_T and is given by

$$M_{t,\tau} = e^{-r\tau}\frac{q_t(S_T)}{p_t(S_T)}$$

where q_t, p_t are conditional densities of $S_T = \exp\left\{\left(\cdot - \frac{\sigma^2}{2}\right)\tau + \log S_t + \sigma(B_T - B_t)\right\}$

Merton Optimization Problem

Market completness, representative investor, $t \leq s \leq T$

1. utility function U
2. wealth proces $\{W_s\}$
3. consumption process $\{C_s\}$, $C_s = 0$
4. consumes all wealth at T, $C_T = W_T$
5. amount $\{\xi_s\}$ invested in S_s

$$\max_{\{\xi_s\}} E[U(W_T)|\mathcal{F}_t]$$

subject to

$$W_s \geq 0$$

$$dW_s = \{rW_s + \xi_s(\mu - r)\}ds + \xi_s \sigma dB_s$$

1. in equilibrium $W_s = S_s$ and

$$e^{-r\tau}\frac{\zeta_s}{\zeta_t} = \frac{J_W(S_s, s)}{J_W(S_t, t)} \quad (1)$$

2. at the end consume all wealth, i.e. $C_T = W_T = S_T$ and

$$e^{-r\tau}\frac{\zeta_T}{\zeta_t} = \frac{U'(W_T)}{U'(W_t)} \quad (2)$$

Merton: Pricing Kernels and Preferences

In Merton asset-pricing model the pricing kernel

1. is path independent (1), can be written as ratio of conditional densities

$$e^{-r\tau}\frac{\zeta_T}{\zeta_t}(S_T, S_t) = e^{-r\tau}\frac{q_t(S_T)}{p_t(S_T)}$$

2. is equal to the marginal rate of substitution (2),

$$e^{-r\tau}\frac{\zeta_T}{\zeta_t}(S_T, S_t) = \frac{U'(S_T)}{U'(S_t)}$$

Thus, it holds

$$
\begin{aligned}
\frac{U'(S_T)}{U'(S_t)} &= e^{-r\tau}\frac{q_t(S_T)}{p_t(S_T)} \\
U(S_T) &= e^{-r\tau}U'(S_t)\int \frac{q_t(S_T)}{p_t(S_T)}dS_T \\
\rho(S_T) &= -S_T\frac{U''(S_T)}{U'(S_T)} \\
&= S_T\left\{\frac{p_t'(S_T)}{p_t(S_T)} - \frac{q_t'(S_T)}{q_t(S_T)}\right\}
\end{aligned}
$$

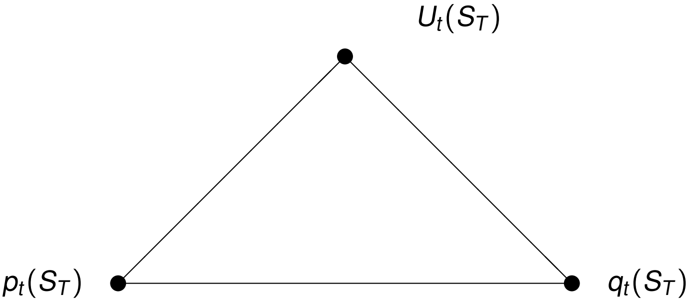


Figure 6: Pricing kernel, utility function, risk neutral and objective measures.

Pricing Kernel Estimation

Empirical pricing kernel $\widehat{M}_t(\kappa, \tau)$

$$\widehat{M}_t(\kappa, \tau) = e^{-r_t \tau} \frac{\hat{q}_t(\kappa, \tau)}{\hat{p}_t(\kappa, \tau)}$$

Ait-Sahalia and Lo (2000)
Estimate state-price density $\hat{q}$ from option prices

Breeden and Lietzenberger (1978)

$$q_t(S_T) = \left. \frac{\partial^2 C_t(S_t, K, \tau, r_t, \sigma_t)}{\partial K^2} \right|_{K=S_T}$$

Ait-Sahalia and Lo (1998)

1. estimated call price function

$$\widehat{C}_t = C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\kappa, \tau)\}$$

   $\widehat{\sigma}_t(\kappa, \tau)$: nonparametric estimator for implied volatility, $C_{t,BS}$: Black-Scholes price at t

2. implied *state-price density*

$$\widehat{q}_t(S_T) = \left. \frac{\partial^2 C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\kappa, \tau)\}}{\partial K^2} \right|_{K=S_T} \tag{3}$$

Degenerated Design

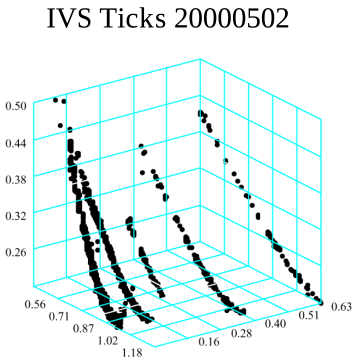
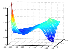


Figure 7: Left panel: call and put implied volatilities observed on 20000502. Right panel: data design on 20000502; ODAX, difference-dividend correction according to Hafner and Wallmeier (2001) applied.



Dynamic semiparametric factor models (DSFM)

$$Y_{i,j} = \sum_{l=0}^{L} z_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}$$

1. $Y_{i,j} = \log \sigma_{i,j}$
2. $\sigma_{i,j}$ implied volatility at trade j on trading day i, $i = 1, \ldots, I$, $j = 1, \ldots, J_i$
3. $m_l(\cdot)$ are basis functions, $l = 0, \ldots, L$, in covariables $X_{i,j}$
4. $z_{i,l}$ are time dependent factors

$X_{i,j} = (\kappa_{i,j}, \tau_{i,j})^\top$ is a two-dimensional vector containing

1. time to maturity $\tau_{i,j}$
2. forward moneyness $\kappa_{i,j} = \frac{K}{F_{i,j}}$

where K is strike and $F_{i,j}$ are futures price

$$F_{i,j} = S_{i,j} \exp(r_{\tau_{i,j}} \tau_{i,j})$$

Following Borak et al. (2007), the basis functions are expanded using a series estimator

$$m_l(X_{i,j}) = \sum_{k=1}^{K} \gamma_{l,k} \psi_k(X_{i,j})$$

for functions $\psi_k : \mathbb{R} \rightarrow \mathbb{R}$, $k = 1, \ldots, K$ and coefficients $\gamma_{l,k} \in \mathbb{R}$

Defining $Z = (z_{i,l})$, $\Gamma = (\gamma_{l,k})$ we obtain the least square estimators

$$(\widehat{\Gamma}, \widehat{Z}) = \arg \min_{\Gamma \in \mathcal{G}, Z \in \mathcal{Z}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left\{ Y_{i,j} - z_i^\top \Gamma \psi(X_{i,j}) \right\}^2$$

where

1. $z_i = (z_{i,0}, \ldots, z_{i,L})^\top$
2. $\psi(x) = \{\psi_1(x), \ldots, \psi_K(x)\}^\top$
3. $\mathcal{G} = \mathcal{M}(L+1, K)$
4. $\mathcal{Z} = \{Z \in \mathcal{M}(I, L+1) : z_{i,0} \equiv 1\}$

Implied volatility and DSFM

The implied volatility at time *i* is estimated as

$$\widehat{\sigma}_i(\kappa, \tau) = exp\left\{\widehat{z}_i^\top \widehat{m}(\kappa, \tau)\right\} \tag{4}$$

where $m(x) = \{m_1(x), \ldots, m_k(x)\}^\top$ and $\widehat{m}_l(x) = \widehat{\gamma}_l^\top \psi(x)$

Implied SPD and DSFM

Combining (3) and (4) implied SPD estimated as

$$\begin{aligned}
\widehat{q}_t(S_T) &= \left.\frac{\partial^2 C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\kappa,\tau)\}}{\partial K^2}\right|_{K=S_T} \\
&= \phi(d_2)\left.\left\{\frac{1}{K\widehat{\sigma}_t\sqrt{\tau}} + \frac{2d_1}{\widehat{\sigma}_t}\frac{\partial\widehat{\sigma}_t}{\partial K} + \frac{K\sqrt{\tau}d_1 d_2}{\widehat{\sigma}_t}\left(\frac{\partial\widehat{\sigma}_t}{\partial K}\right)^2 + K\sqrt{\tau}\frac{\partial^2\widehat{\sigma}_t}{\partial K^2}\right\}\right|_{K=S_T}
\end{aligned}$$

where $\phi(x)$ is pdf from standard normal distribution

Empirical Results

Intraday DAX index and option data

1. from 20010101 to 20020101
2. 253 trading days
3. model selection: $L = 3$
4. $\hat{q}_t$ estimated with DSFM
5. $\hat{p}_t$ estimated from last 240 days with GARCH(1,1)

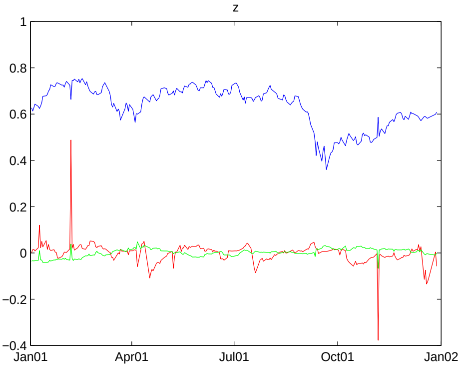


Figure 8: Loading factors $\hat{z}_{tl}$, $l = 1, 2, 3$ from the top

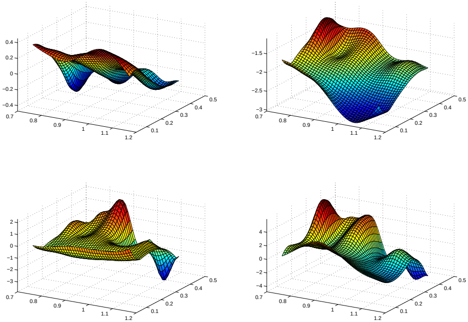


Figure 9: Basis functions $\hat{m}_l$, $l = 0, \ldots, 3$ clockwise

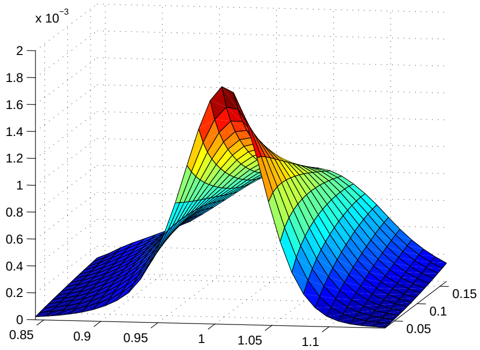


Figure 10: Estimated SPD across κ and τ at $t = 20010710$

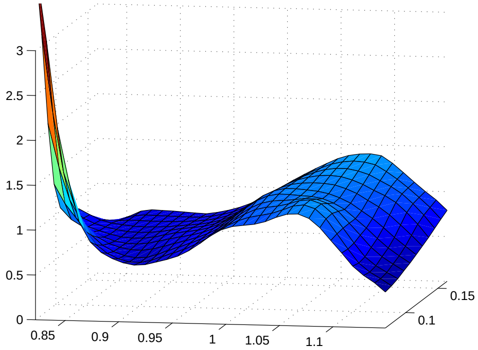


Figure 11: Estimated PK across κ and τ at $t = 20010710$

Pricing Kernel and SPD dynamics

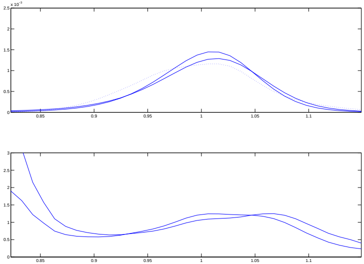


Figure 12: $\widehat{p}_{t_0}$ (dash), $\widehat{q}_{t_0}$, $\widehat{q}_{t_1}$ (top), $\frac{\widehat{q}_{t_0}}{\widehat{p}_{t_0}}$, $\frac{\widehat{q}_{t_1}}{\widehat{p}_{t_0}}$, $t_0 = 20010710$, $t_1 = 20010730$

Pricing Kernel and SPD dynamics

1. influence of loading factors in q

$$\hat{q}_t(S_T) \quad = \quad \phi(d_2)\left\{\frac{1}{K\hat{\sigma}_t\sqrt{\tau}} + \frac{2d_1}{\hat{\sigma}_t}\frac{\partial\hat{\sigma}_t}{\partial K} + \frac{K\sqrt{\tau}d_1 d_2}{\hat{\sigma}_t}\left(\frac{\partial\hat{\sigma}_t}{\partial K}\right)^2 + K\sqrt{\tau}\frac{\partial^2\hat{\sigma}_t}{\partial K^2}\right\}\Bigg|_{K=S_T}$$

2. analyse first moment of q across time and maturities

$$\mu_{t,\tau} = \frac{1}{S_t e^{r\tau}} \int x\widehat{q}_{t,\tau}(x)dx$$

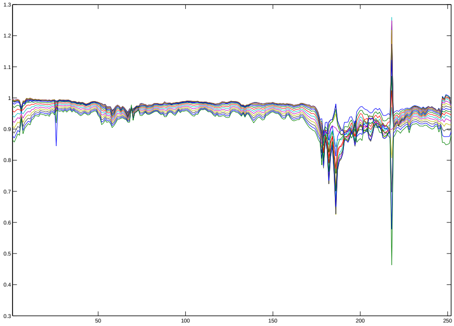


Figure 13: $\mu_{t,\tau}$, $\tau = 0.06, \ldots, 2.21$.

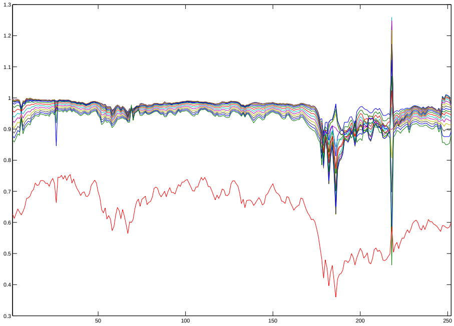


Figure 14: $\mu_{t,\tau}$, $\tau = 0.06\ldots,2.21$, $\widehat{z}_1$ (below).

Sensitivity to Loading Factors

One increasing factor, remaining factors constant at sample median, $n = 0, \ldots, N$, $l, k = 1, \ldots, 3$ and $l \neq k$,

$$
\begin{aligned}
z^*_{ln} &= d_l + \frac{n}{N}(u_l - d_l) \\
z_{kn} &\equiv med(z_k) \\
d_l &= \min \hat{z}_{tl} - 0.5|\min \hat{z}_{tl}| \\
u_l &= \max \hat{z}_{tl} + 0.5|\max \hat{z}_{tl}|
\end{aligned}
$$

	z_{t1}	z_{t2}	z_{t3}
min	0.36	-0.37	-0.07
max	0.75	0.49	0.05
median	0.66	0.01	0.00
mean	0.63	0.00	0.00
std.dev.	0.09	0.05	0.02
u	1.13	0.73	0.07
d	0.18	-0.57	-0.10

Table 1: Descriptive statistics of loading factors.

First factor loading $\widehat{z}_1$

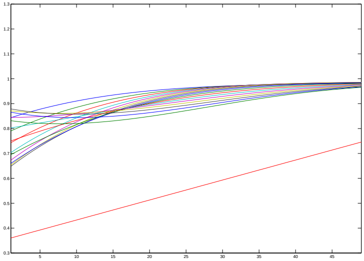


Figure 15: $\mu_{t,\tau}$ estimated with z^*_{1n}, $n = 0, \ldots, 50$, $\tau = 0.06, \ldots, 2.21$ (below).

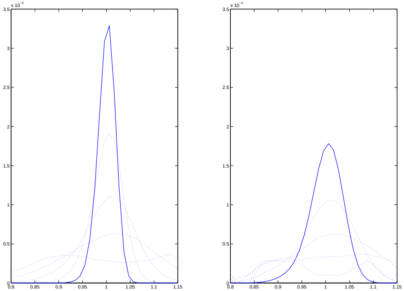


Figure 16: SPD estimated with z^*_{1n}, $n = 0, \ldots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

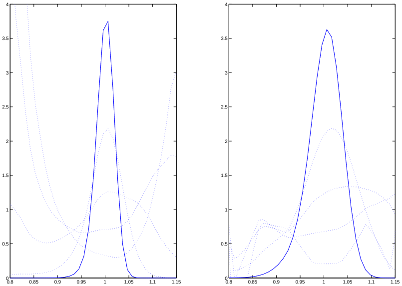


Figure 17: PK estimated with z^*_{1n}, $n = 0, \dots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

Second factor loading $\widehat{z}_2$

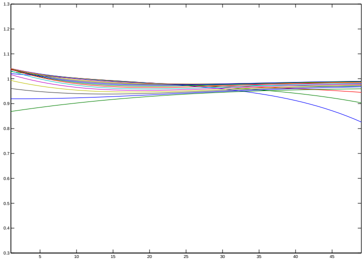


Figure 18: $\mu_{t,\tau}$ estimated with z^*_{2n}, $n = 0, \ldots, 50$, $\tau = 0.06, \ldots, 2.21$.

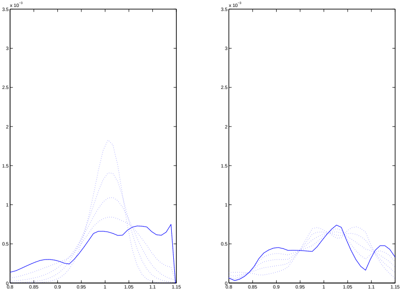


Figure 19: SPD estimated with z^*_{2n}, $n = 0, \ldots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

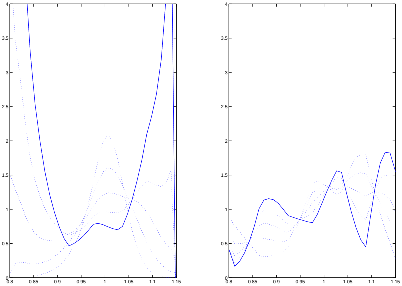


Figure 20: PK estimated with z^*_{2n}, $n = 0, \ldots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

Third factor loading $\widehat{z}_3$

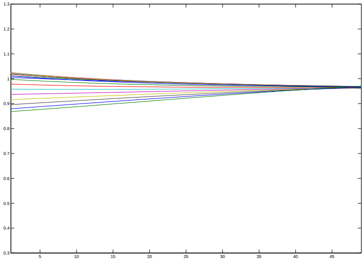


Figure 21: $\mu_{t,\tau}$ estimated with z^*_{3n}, $n = 0, \ldots, 50$, $\tau = 0.06, \ldots, 2.21$ (below).

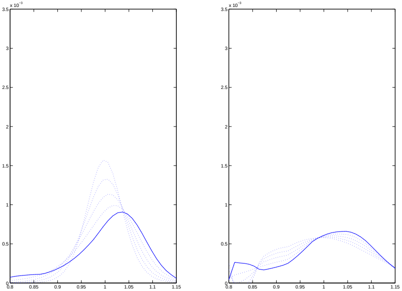


Figure 22: SPD estimated with z^*_{3n}, $n = 0, \ldots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

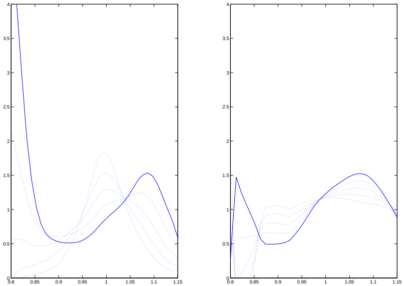


Figure 23: PK estimated with z^*_{3n}, $n = 0, \ldots, 3$ (dash), $N = 4$ (solid). Left $\tau = 0.06$, right $\tau = 0.21$

Outlook

1. influence of remaining loading factors on SPD
2. correlation between loading factors
3. correlation between moments and loading factors

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From (1) ζ_t is path independent and

$$
\begin{aligned}
\zeta_t &= \frac{q(S_0, S_t)}{p(S_0, S_t)} \\
&= \frac{q(s_0, s_{0<k\leq s}, S_{s<k\leq t})}{p(s_0, s_{0<k\leq s}, S_{s<k\leq t})}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\zeta_s}{\zeta_t} &= \frac{q(s_0, s_{0<k\leq t}, S_{t<k\leq s})}{q(s_0, s_{0<k\leq t})} \frac{p(s_0, s_{0<k\leq t})}{p(s_0, s_{0<k\leq t}, S_{t<k\leq s})} \\
&= \frac{q_s(S_t)}{p_s(S_t)}
\end{aligned}
$$

where $q_t(S_s)$ and $p_t(S_s)$ are conditional densities from S_s at time t, $s > t$