# Applications of Copulae for the Calculation of Value-at-Risk

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# Copula vs Normal Distribution

1. The empirical marginal distributions are skewed and fat tailed.
2. Multivariate normal distribution does not consider the possibility of extreme joint co-movement of asset returns. The dependency structure of portfolio asset returns is different from the Gaussian one.

# Advantages

1. Copulae are useful tools to simulate asset return distributions in a more realistic way.
2. Copulae allow to model the dependence structure independently from the marginal distributions
   - construct a multivariate distribution with different margins
   - the dependence structure is given by the copula.

# Dependency Structures

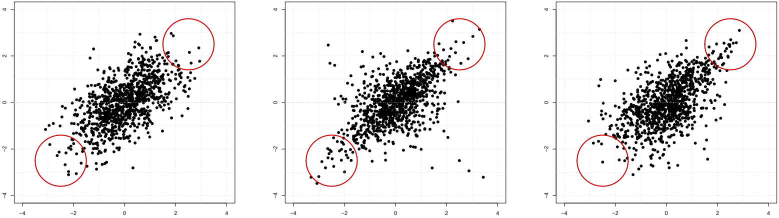



Figure 1: Scatter plots of bivariate samples with different dependency structures and equal correlation coefficient.

# Varying Dependency

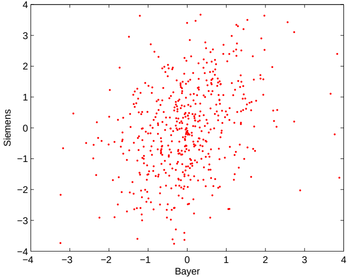



Figure 2: Standardized log returns of Bayer and Siemens 20000103-20020101 (left) and 20040101-20060102 (right).

# Outline

# Copulae

A copula is a multivariate distribution function defined on the unit cube $[0,1]^d$, with uniformly distributed margins.

$$\begin{aligned} P(X_1 \leq x_1, \ldots, X_n \leq x_d) &= C\left\{P(X_1 \leq x_1), \ldots, P(X_d \leq x_d)\right\} \\ &= C\left\{F_1(x_1), \ldots, F_d(x_d)\right\} \end{aligned}$$

# Bivariate Copulae

A *2-dimensional copula* is a function $C : [0,1]^2 \rightarrow [0,1]$ with the following properties:

1. For every $u \in [0,1]$, $C(0,u) = C(u,0) = 0$ (**grounded**)
2. For every $u \in [0,1]$, $C(u,1) = u$ and $C(1,u) = u$
3. For every $(u_1,u_2),(v_1,v_2) \in [0,1] \times [0,1]$ with $u_1 \leq v_1$ and $u_2 \leq v_2$: $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \geq 0$ (**2-increasing**)

# Multivariate Copula

A *d-dimensional copula* is a function $C : [0,1]^d \rightarrow [0,1]$:

1. $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_d) = 0$ (at least one $u_i$ is 0);
2. $u \in [0,1]$, $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ (all coordinates except $u_i$ is 1)
3. For each $u < v \in [0,1]^d$ $(u_i < v_i)$

$$V_C[u,v] = \sum_a sgn(a) C(a) \geq 0$$

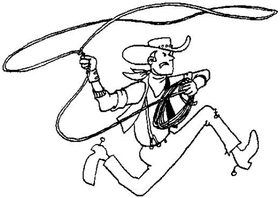
   where $a$ is taken over all vertices of $[u,v]$. $sgn(a) = 1$ if $a_k = u_k$ for an even number of $k'$s and $sgn(a) = -1$ if $a_k = u_k$ for an odd number of $k'$s (**d-increasing**)

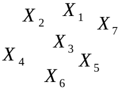
# Sklar's Theorem

For a distribution function $F$ with marginals $F_{X_1} \dots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \tag{1}$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \dots, d$. If $F_{X_1}, \dots, F_{X_d}$ are cts, then $C$ is unique. If $C$ is a copula and $F_{X_1}, \dots, F_{X_d}$ are cdfs, then the function $F$ defined in (1) is a joint cdf with marginals $F_{X_1}, \dots, F_{X_d}$.





- a copula $C$ and marginal distributions can be "coupled" together into a distribution function:

$$F_X(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\}$$

- a (unique) copula is obtained from "decoupling" every (continuous) multivariate distribution function from its marginal distributions:

$$C(u_1, \dots, u_d) = F_X\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\}$$

$$u_j = F_{X_j}(x_j), \quad j = 1, \dots, d$$

- if $C$ is absolute continuous there exists a copula density

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}$$

- the joint density $f_X$ is

$$f_X(x_1, \dots, x_d) = c\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \prod_{j=1}^{d} f_j(x_j)$$

# Fréchet-Hoeffding Bounds, Product Copula

1. every copula $C$ satisfies

$$W(u_1, \ldots, u_d) \leq C(u_1, \ldots, u_d) \leq M(u_1, \ldots, u_d)$$

2. upper and lower bounds

$$\begin{aligned} M(u_1, \ldots, u_d) &= \min(u_1, \ldots, u_d) \\ W(u_1, \ldots, u_d) &= \max\left(\sum_{i=1}^{d} u_i - d + 1, 0\right) \end{aligned}$$

3. product copula

$$\Pi(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$$

# Fréchet Copulae

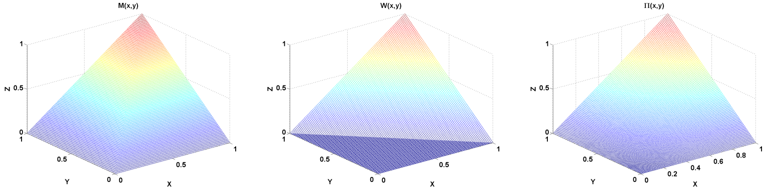



Figure 3: $M(u,v) = \min(u,v)$, $W(u,v) = \max(u+v-1, 0)$ and $\Pi(u,v) = uv$

*M. Fréchet* on BBI: 

# Product Copula

Let $X_1$ and $X_2$ be random variables with continuous distribution functions $F_1$ and $F_2$ and joint distribution function $H$.
Then $X_1$ and $X_2$ are independent if and only if $C_{X_1 X_2} = \Pi$.
According to Sklar's Theorem, there exists a unique copula $C$ with

$$\begin{aligned} P(X_1 \leq x_1, X_2 \leq x_2) &= H(x_1, x_2) \\ &= C\left\{F_1(x_1), F_2(x_2)\right\} \\ &= F_1(x_1) \cdot F_2(x_2) \end{aligned}$$

# Gauss Copula

$$
\begin{aligned}
C(u_1, u_2) &= \Phi_\rho\{\Phi^{-1}(u_1), \Phi^{-1}(u_2)\} \\
&= \int\limits_{-\infty}^{\Phi^{-1}(u_1)} \int\limits_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dx\, dy
\end{aligned}
$$

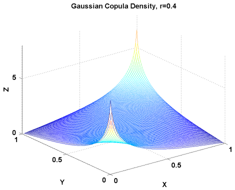



Figure 4: Gauss copula density, parameter $\rho = 0.4$.

*C. Gauss* on BBI: 

# $t$-Student Copula

$$
\begin{aligned}
C(u_1,u_2) &= t_{\rho,\nu}\{t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\}\\
&= \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{1+\frac{x^2-2\rho xy+y^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2} dx\,dy
\end{aligned}
$$

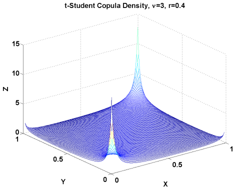



Figure 5: $t$-Student copula density, parameters $\nu = 3$ and $\rho = 0.4$.

*W. Gosset* on BBI: 

# Archimedean Copulae

Archimedean copula:

$$C(u,v) = \psi^{[-1]}\{\psi(u) + \psi(v)\}$$

for a continuous, decreasing and convex $\psi$, $\psi(1) = 0$.

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t), & 0 \le t \le \psi(0), \\ 0, & \psi(0) < t \le \infty. \end{cases}$$

The function $\psi$ is a generator of the Archimedean copula.
For $\psi(0) = \infty$: $\psi^{[-1]} = \psi^{-1}$ and the $\psi$ is called a strict generator.

# Gumbel Copula

$$C(u,v) = \exp\left[ - \left\{ (-\log u)^{\theta} + (-\log v)^{\theta} \right\}^{\frac{1}{\theta}} \right]$$

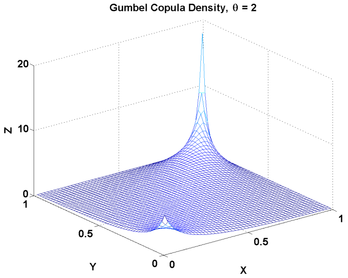



Figure 6: Gumbel copula density, parameter $\theta = 2$.

*E. Gumbel* on BBI: 

# Clayton Copula

$$C(u,v) = \max\left\{(u^{-\theta} + v^{-\theta} - 1)^{\frac{1}{\theta}}, 0\right\}$$

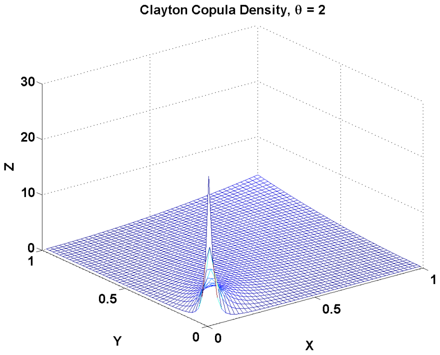



Figure 7: Clayton copula density, parameter $\theta = 2$.

# Frank Copula

$$C(u,v) = -\frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right\}$$

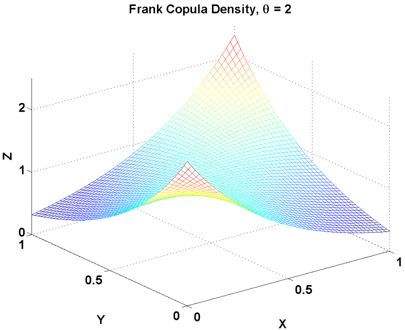



Figure 8: Frank copula density, parameter $\theta = 2$.

# Multivariate Elliptical Copulae

- Gauss

$$\int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} |R|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} r^\top R^{-1} r\right) \, dr_1 \ldots dr_d,$$

  where $r = (r_1, \ldots, r_n)^\top$

- $t$-Student

$$\int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} |R|^{-\frac{1}{2}} \left(1 + \frac{r^\top R^{-1} r}{\nu}\right)^{-\frac{v+n}{2}} \, dr_1 \ldots dr_d$$

  where $r = (r_1, \ldots, r_n)^\top$

# Multivariate Archimedean Copulae

- Gumbel

$$C(u_1, \dots, u_d) = \exp\left[ - \left\{ (-\log u_1)^\theta + \dots + (-\log u_d)^\theta \right\}^{\frac{1}{\theta}} \right]$$

- Cook-Johnson

$$C(u_1, \dots, u_d) = \left( \sum_{j=1}^{n} u_j^{-\theta} - d + 1 \right)^{-\frac{1}{\theta}}$$

- Frank

$$C(u_1, \dots, u_d) = -\frac{1}{\theta} \log\left\{ 1 + \frac{(e^{-\theta u_1} - 1) \dots (e^{-\theta u_d} - 1)}{(e^{-\theta} - 1)^{d-1}} \right\}$$

# Dimensionality

In $d$-dimension

1. Elliptical Copulae: correlation matrix with $\frac{d(d-1)}{2}$ parameters
2. Archimedean Copulae: 1 parameter

# Parameter Estimation

- Full Maximum Likelihood (FML)
- Method of Inference Functions for Margins (IFM)
- Canonical Maximum Likelihood (CML) method

# Copula Estimation

Given observations $\{x_t\}_{t=1}^{T}$ the log-likelihood function for a copula $C_\theta$, marginal distributions $F_j(x_j)$ and parameters $\alpha = (\delta_1, \ldots, \delta_d, \theta)^\top$ is

$$\ell(\alpha; x_1, \ldots, x_T) =$$

$$= \sum_{t=1}^{T} \log\, c\{F_{X_1}(x_{1,t}; \delta_1), \ldots, F_{X_d}(x_{d,t}; \delta_d); \theta\} + \sum_{t=1}^{T} \sum_{j=1}^{d} \log\, f_j(x_{j,t}; \delta_j)$$

# Full Maximum Likelihood - FML

The parameters are estimated through

$$\tilde{\alpha}_{FML} = \arg\max_{\alpha} \ell(\alpha)$$

The estimates $\tilde{\alpha}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve

$$(\partial \ell / \partial \delta_1, \ldots, \partial \ell / \partial \delta_d, \partial \ell / \partial \theta) = 0$$

# Inference Functions for Margins - IFM

**1. step:**
Estimating the parameters $\delta_j$, $j = 1, \ldots, d$ of the marginal distributions $F_{X_j}$ using the ML method

$$\hat{\delta}_j = \arg\max_{\delta_j} \ell_j(\delta_j) = \arg\max_{\delta_j} \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j),$$

where $\ell_j$ is the log-likelihood function of the marginal distribution $F_{X_j}$ with density $f_j$.

# Inference Functions for Margins - IFM

**2. step:**

Estimating the copula parameters $\theta$,

$$\hat{\theta} = \arg\max_{\theta} \ell(\theta) = \arg\max_{\theta} \sum_{t=1}^{T} \log c(F_{X_1}(x_{1,t};\hat{\delta}_1), \ldots, F_{X_d}(x_{d,t};\hat{\delta}_d);\theta),$$

where $\ell$ is the log-likelihood function of the copula.
The estimates $\hat{\alpha}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0$$

# Canonical Maximum Likelihood

In the CML method no assumptions are made about the parametric form of the marginal distributions.
The CML estimator maximizes the pseudo log-likelihood function with empirical marginal distributions $\hat{F}_j$

$$\ell(\theta) = \sum_{t=1}^{T} \log\ c\{\hat{F}_1(x_1), \ldots, \hat{F}_d(x_d); \theta\}$$

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta)$$

where

$$\hat{F}_j(x) = \frac{1}{T+1} \sum_{t=1}^{T} I(X_{j,t} \leq x), j = 1, \ldots, d$$

# Multivariate Gaussian Copula

Algorithm of simulating pseudo rvs from Gaussian copula with correlation matrix $R$

1. Perform a Cholesky decomposition $R = A^\top A$.
2. Simulate $n$ independent rvs $\mathbf{z} = z_1, \ldots, z_n$ from $N(0,1)$.
3. Set $(x_1, \ldots, x_n) = \mathbf{x} = A\mathbf{z}$.
4. Set $u_i = \Phi(x_i)$, $i = 1, \ldots, n$.

$(u_1, \ldots, u_n)^\top \sim C_R^{Gauss}$.

# Multivariate $t$-Student

Algorithm of simulating pseudo rvs from $t$-Student copula with correlation matrix $R$ and $\nu$ degrees of freedom

1. Perform a Cholesky decomposition $R = A^\top A$.
2. Simulate $n$ independent rvs $\mathbf{z} = z_1, \ldots, z_n$ from $N(0,1)$.
3. Simulate a random variate $s$ from $\chi^2_\nu$ independent of $\mathbf{z}$.
4. Set $(y_1, \ldots, y_n) = \mathbf{y} = A\mathbf{z}$.
5. Set $\mathbf{x} = \frac{\sqrt{\nu}}{\sqrt{s}}\mathbf{y}$.
6. Set $u_i = t_\nu(x_i)$, $i = 1, \ldots, n$.

$(u_1, \ldots, u_n)^\top \sim C^t_{\nu,R}$.

# Conditional Inverse Method

The method is based on the conditional distributions of a random vector $\mathbf{U} = (U_1, \ldots, U_d)$.
Let $U_1$, ..., $U_d$ have joint distribution function $C$. Then conditional distribution of $U_k$ given the values of $U_1$, ..., $U_{k-1}$ is given by

$$\begin{aligned}
\Lambda(u_k) &= C(u_k|u_1, \ldots, u_{k-1}) = P(U_k \leq u_k | U_1 = u_1, \ldots, U_{k-1} = u_{k-1}) \\
&= \frac{\frac{\partial^{k-1}}{\partial u_1 \ldots \partial u_{k-1}} C(u_1, \ldots, u_k, 1, \ldots, 1)}{\frac{\partial^{k-1}}{\partial u_1 \ldots \partial u_{k-1}} C(u_1, \ldots, u_{k-1}, 1, \ldots, 1)}.
\end{aligned}$$

# Conditional Inverse Method

The generation follows the steps:

1. generate $v_1, \ldots, v_d$ independent and uniformly distributed in $[0, 1]$.
2. for $n = 1, \ldots, d$ generate $u_n = \Lambda^{-1}(v_n)$.
   $u_1, \ldots, u_d$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by copula $C$.
3. set $x_n = F_n^{-1}(u_n)$.
   $x_1, \ldots, x_d$ have the desired marginal distributions.

# Laplace Transform Archimedean Copulae

The considered copulae – Gumbel, Clayton and Frank – fall into the class of Laplace transform Archimedean copulae.
For this class, the inverse of the generator $\psi$ has a representation of a Laplace transform $\hat{G}$ of some distribution function $G$:

$$\psi^{-1}(t) = \hat{G}(t) = \int_0^{\infty} e^{-tx} dG(x), \quad t \geq 0.$$

We set $\hat{G}(\infty) = 0$.
$\hat{G}(t)$ is continuous and strictly decreasing function.

# Laplace Transform Algorithm (Marshal-Olkin Method)

1. Generate a pseudo rv $V$ with cdf $G$
   - For a Clayton copula, $V$ is gamma distributed, $Ga(\frac{1}{\theta})$, and $\hat{G}(t) = (1+t)^{-1/\theta}$
   - For a Gumbel copula $V$ is stable distributed, $St(\frac{1}{\theta}, 1, \gamma, 0)$ with $\gamma = \{\cos(\frac{\pi}{2\theta})\}^{\theta}$ and $\hat{G}(t) = \exp(-t^{1/\theta})$
   - For a Frank copula, $V$ is discrete with $P(V = k) = (1 - e^{-\theta})^k / (k\theta)$ for $k = 1, 2, \ldots$
2. Generate iid uniform pseudo rvs $X_1, \ldots, X_d$
3. Return $U_i = \hat{G}(-\frac{\ln X_i}{V})$, $i = 1, \ldots, d$.

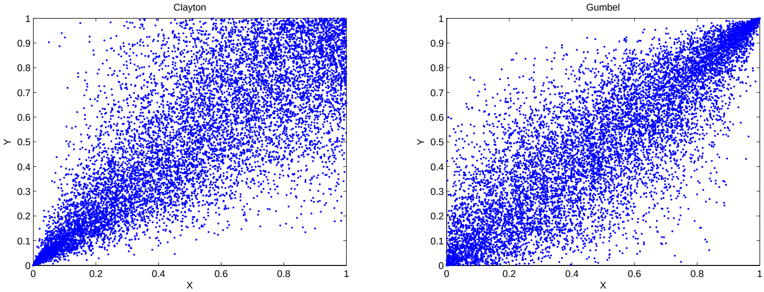



Figure 9: Monte Carlo sample of 10.000 realizations of pseudo random variable with uniform marginals in $[0,1]$ and dependence structure given by Clayton (left) and Gumbel (right) copula with $\theta = 3$.

# Tail Dependence

- Risk behavior is determined by tails large losses that can occur jointly.
- Pearson's correlation can not capture joint large loss events.
- Tail dependence describes the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold.

# Upper tail Dependence

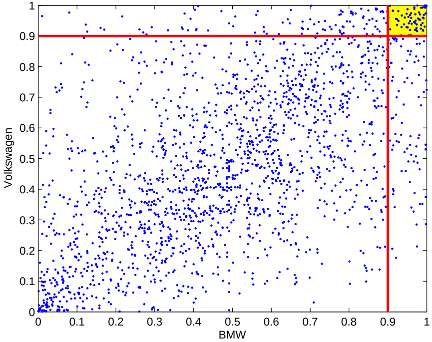



Figure 10: UTD for standardized log-returns of BMW vs Volkswagen transformed by $t$-Student cdf.

# Upper tail Dependence

Let $(X_1, X_2) \sim F$ with margins $F_1$ and $F_2$.
Coefficient of upper tail dependence (UTD):

$$\lambda_U = \lim_{u \nearrow 1} P\{Y > F_2^{-1}(u) | X > F_1^{-1}(u)\}.$$

Asymptotical upper tail dependence: $\lambda_U \in (0, 1]$.
Asymptotical upper tail independence: $\lambda_U = 0$.

# Lower tail dependence

Let $(X_1, X_2) \sim F$ with margins $F_1$ and $F_2$.
Coefficient of lower tail dependence:

$$\lambda_L = \lim_{u \searrow 0} P\{Y \leq F_2^{-1}(u) | X \leq F_1^{-1}(u)\}.$$

Asymptotical lower tail dependence: $\lambda_L \in (0, 1]$.
Asymptotical lower tail independence: $\lambda_U = 0$.

# Tail Dependence and Copulae

Tail dependence is a copula property:

$$
\begin{aligned}
\lambda_U &= \lim_{v \nearrow 1} \frac{1 - 2v + C(v, v)}{1 - v}, \\
\lambda_L &= \lim_{v \searrow 0} \frac{C(v, v)}{v}.
\end{aligned}
\tag{2}
$$

| **Copula** | $\lambda_U$ | $\lambda_L$ |
|---|---|---|
| **Gauss** | 0 for $\rho < 1$ | 0 for $\rho < 1$ |
| | 1 for $\rho = 1$ | 1 for $\rho = 1$ |
| $t_\nu$ | $2\bar{t}_{\nu+1}\left(\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$ | $\lambda_U$ |
| **Gumbel** | $2 - 2^{\frac{1}{\theta}}$ | 0 |
| **Clayton** | 0 | $2^{-\frac{1}{\theta}}$ |
| **Frank** | 0 | 0 |

Table 1: TDCs for various selected copulae.

# Risk Measures

1. Value-at-Risk (negative)

$$VaR^X_{1-\alpha} = Q^X_\alpha = -q^{-X}_{1-\alpha},$$

   - $Q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) > \alpha\}$,
   - $q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$.

2. Expected Shortfall

$$ES^X_{1-\alpha} = E(X | X < VaR^X_{1-\alpha}).$$

# Value-at-Risk with Copulae

For a sample of log-returns $\{X_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$

1. specification of marginal distributions $F_{X_j}(x_j; \delta_j)$
2. specification of copula $C(u_1, \ldots, u_d; \theta)$ where $u_j = F_{X_j}(x_j)$
3. fit of the copula $C$ (estimation the copula parameters)
4. generation of $n$ Monte Carlo data $U_{T+1} \sim C\{F_1(x_1), \ldots, F_d(x_d); \hat{\theta}\}$
5. generation of a sample of portfolio profits $L_{T+1}(X_{T+1})$
6. estimation of $\widehat{VaR}_{1-\alpha}$, the empirical quantile from $L_{T+1}$.

# Estimation of VaR

$$\widehat{VaR}^{L}_{1-\alpha} = L_{(\lfloor \alpha n \rfloor + 1):n}$$

where $L$ is Profit and Loss function

$$\begin{aligned} L_{t+1} &= \sum_{j=1}^{d} S_{j,t+1} - \sum_{j=1}^{d} S_{j,t} \\ &= \sum_{j=1}^{d} S_{j,t}(\exp(X_{j,t+1}) - 1) \end{aligned}$$

and $X_{t+1} = \log S_{t+1} - \log S_t$.

# Generation of Possible Scenarios

Assume that the standardized returns of margin $j$, $j = 1, \ldots, d$, are modeled with $t$-Student distribution with $\nu_j$ degrees of freedom. Generation of possible values of change of the portfolio at time $T+1$ follows the steps:

1. sampling $n = 10.000$ pseudo rvs for each $U_{1,T+1}, \ldots, U_{d,T+1}$
2. generation $t$-distributed rvs by $V_{j,T+1} = \text{tinv}(U_{j,T+1}, \nu_j)$
3. generation of the values of possible log-returns
   $X_{j,T+1} = V_{j,T+1} \cdot \text{std}_j + \text{mean}_j$
4. determinig values of profit and loss function
   $L_{T+1} = \sum_{j=1}^{d} S_{j,T}(\exp(X_{j,T+1}) - 1)$

# Moving Window

- Specify the subsets of size $h = 250$: $\{u_{j,t}\}_{t=s-h+1}^{s}$ for $s = h, \ldots, T$.
- Obtain the sequence $\{\widehat{VaR}_{1-\alpha}^{j}\}_{j=1}^{T-h}$ and $\{\theta_j\}_{j=1}^{T-h}$.

# Moving window

For a sample of log-returns $\{X_t\}_{t=1}^{T}$

1. specification of marginal distributions $F_{X_j}(x_j; \delta_j)$
2. specification of returns' subsets of size $h$: $\{y_{j,t}\}_{t=s-h+1}^{s}$ for $s = h, \ldots, T-1$
3. specification of copulae $C_s(u_1, \ldots, u_d; \theta)$ for every subset $\{y_{j,t}\}_{t=s-h+1}^{s}$
4. fit of the copulae $C_s$, $s = h, \ldots, T-1$
5. generation of Monte Carlo data $U_{s+1} \sim C_s\{F_1(x_1), \ldots, F_d(x_d); \hat{\theta}\}$ for $s = h, \ldots, T-1$
6. generation of a samples of portfolio profits $L_{s+1}(X_{s+1})$
7. estimation of $\{\widehat{VaR}_{1-\alpha}^{j}\}_{j=1}^{T-h}$.

# Backtesting

The estimated VaR values are compared with true realizations $\{L_t\}$ of the Profit and Loss function.
An *exceedance* occurs when $L_t$ is smaller than $\widehat{VaR}^t_{1-\alpha}$.
The ratio of the number of exceedances to the number of observations gives the *exceedances ratio*:

$$\hat{p} = \frac{1}{T-h} \sum_{t=h+1}^{T} I_{\{L_t < \widehat{VaR}^t_{1-\alpha}\}}.$$

# Application

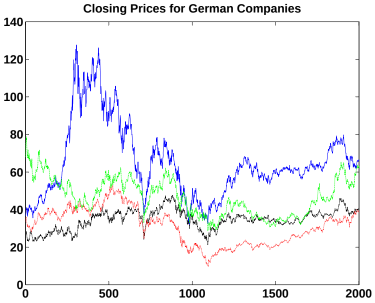



Figure 11: Closing prices of stocks: BMW, Bayer, Siemens, Volkswagen. Time period: 1st January 1999 – 1st September 2006, 2000 data points.

# Returns

Let $P_1, \ldots, P_n$ be a time series of stock's prices.

- Simple return is defined as

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

- Logarithmic return (log-return) is defined as

$$r_t = \log \frac{P_t}{P_{t-1}}.$$

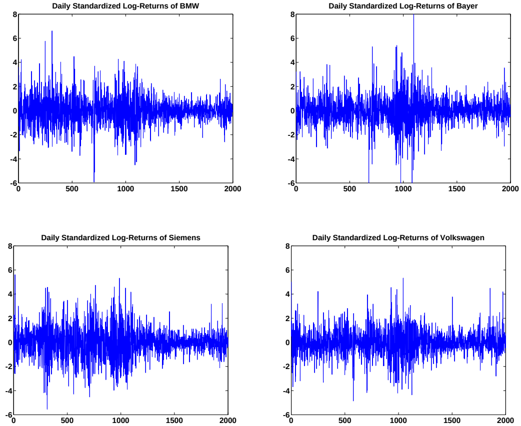



Figure 12: Daily stock standardized log-returns: BMW, Bayer, Siemens, Volkswagen.

# Margins

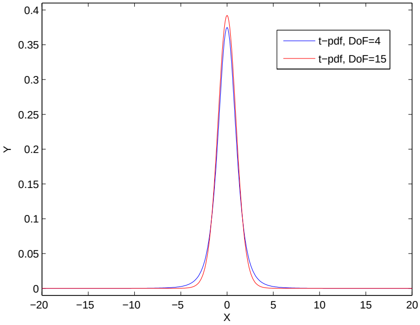



Figure 13: Standardized margins are modeled with $t$-Student distribution with degrees of freedom equal 7 for BMW, 6 for Bayer, 5 for Siemens, 8 for Volkswagen.

# Value-at-Risk Estimation

| Copula | BAY - SIE | BMW - VOW | SIE - VOW |
|---|---|---|---|
| Gauss | 0.0320 | 0.0394 | 0.0366 |
| $t$-Student | 0.0314 | 0.0405 | 0.0371 |
| Gumbel | 0.0360 | 0.0400 | 0.0394 |
| Clayton | 0.0308 | 0.0348 | 0.0354 |
| Frank | 0.0337 | 0.0400 | 0.0366 |
| Normal distribution | 0.1216 | 0.0999 | 0.1182 |

Table 2: Backtesting results for Value-at-Risk estimation at 0.05 level for 3 portfolios, $w = (1,1)^T$, size of moving window 250, Monte Carlo samples of 10.000 realizations of pseudo random variable. Standardized margins modeled with $t$-distribution.

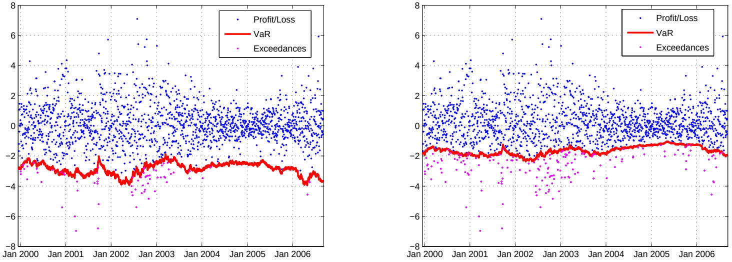



Figure 14: VaR, P&L and exceedances estimated with $t$-Student copula ($\hat{\alpha} = 0.0405$) and bivariate normal distribution ($\hat{\alpha} = 0.0999$) for BMW and Volkswagen.

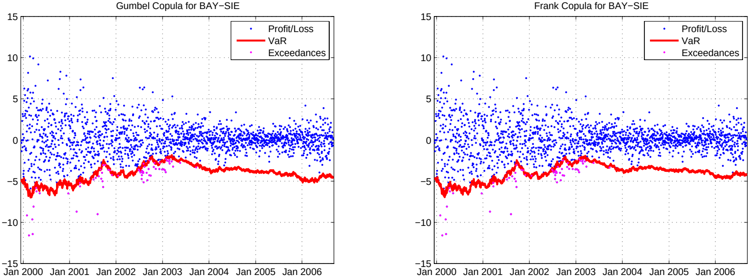



Figure 15: VaR, P&L and exceedances estimated with Gumbel copula ($\hat{\alpha} = 0.0360$) and Clayton copula ($\hat{\alpha} = 0.0308$) for Bayer and Siemens.

# Conclusions

**Pluses of copulae**

- flexible and wide range of dependence
- easy to simulate, estimate, implement
- explicit form of densities of copulae
- modelling of fat tails, assymetries

**Minuses of copulae**

- Elliptical: correlation matrix, symmetry
- Archimedean: too restrictive, single parameter, exchangable
- selection of copula

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