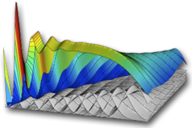
Volatility Investing with Variance Swaps

Wolfgang Karl Härdle
Elena Silyakova

Ladislaus von Bortkiewicz
Chair of Statistics
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de



Why investors may wish to trade volatility?

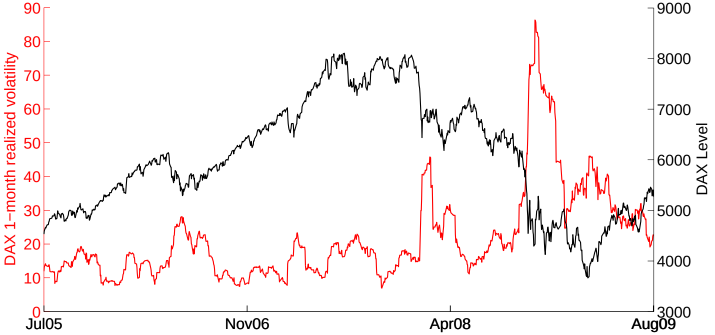


Figure 1: DAX level vs. DAX 1M realized volatility (20050103 - 20091230)

Research questions

- How to trade volatility?
- How to hedge (replicate) volatility?
- How good can we perform?

Outline

1. Motivation ✓
2. Definition
3. Replication and hedging
4. 3G volatility derivatives
5. Dispersion trading strategy
6. Conclusions

Variance swap

Figure 2: Cash flow of a variance swap at expiry

Definition

Variance swap is a forward contract that at maturity pays the difference between realized variance σ_R^2 and predefined strike K_{var}^2 multiplied by notional N_{var}.

$$(\sigma_R^2 - K_{var}^2) \cdot N_{var} \tag{1}$$

$$\sigma_R = \sqrt{\frac{252}{T} \sum_{t=1}^{T} \left(\log \frac{S_t}{S_{t-1}} \right)^2} \cdot 100 \tag{2}$$

Construction

Example: 3-month variance swap long

Long position in 3-month variance swap. Trade size is 2500 variance notional (represents a payoff of 2500 per point difference between realized and implied variance).

If K_{var} is 20% ($K_{var}^2 = 400$) and the subsequent variance realized over the course of the year is $(15\%)^2$ (quoted as $\sigma_R^2 = 225$), the investor will make a loss because realized variance is below the level bought.

Overall loss: $437500 = 2500 \cdot (400 - 225)$

Replication and hedging - intuitive approach

- consider a European option with Black-Scholes (BS) price $V_{BS}(S, K, \sigma\sqrt{\tau})$
- variance vega:

$$\frac{\partial V_{BS}}{\partial \sigma^2} = \frac{S}{2\sigma\sqrt{\tau}}\varphi(y) \tag{3}$$

where

$$y = \frac{\log(S/K) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$$

φ - pdf of a standard normal rv.

Variance vega of options with different K

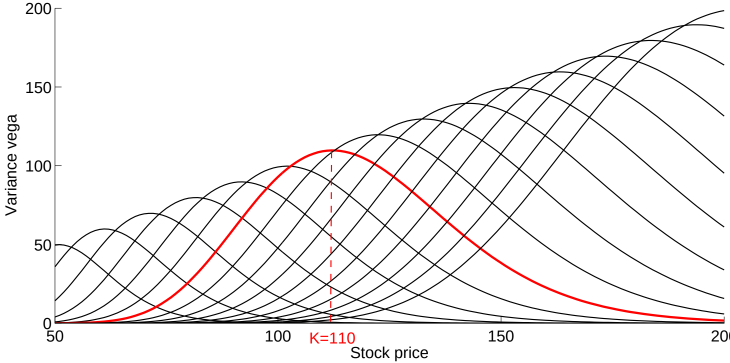


Figure 3: Dependence of variance on S for vanilla options with $K = [50, 200]$, $\sigma = 0.2$, $\tau = 1$

Equally-weighted option portfolio

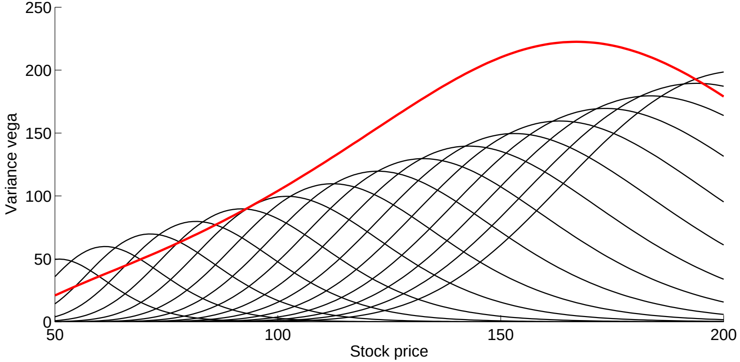


Figure 4: Variance vega of option portfolio (red line) with options weighted equally

$1/K$-weighted option portfolio

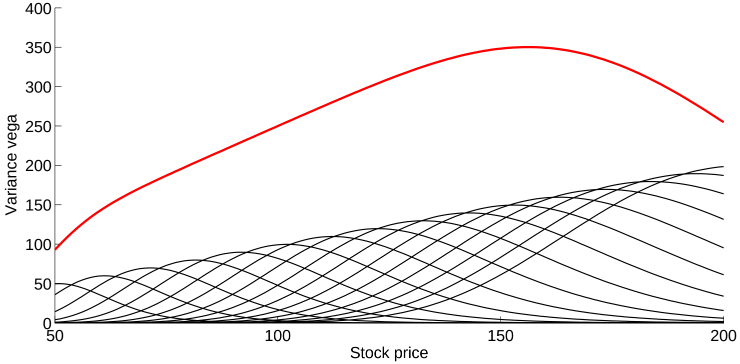


Figure 5: Variance vega of option portfolio (red line) with options weighted proportional to $1/K$

$1/K^2$-weighted option portfolio

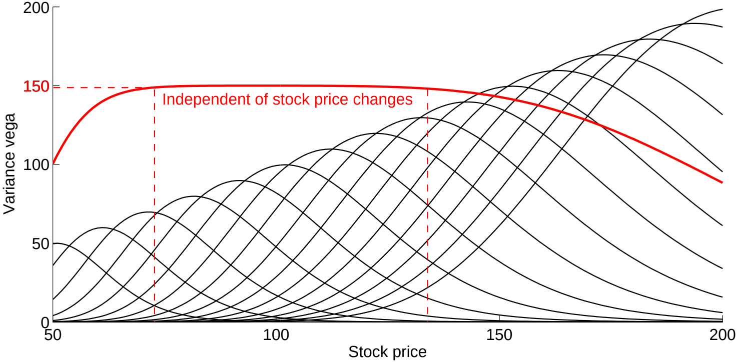


Figure 6: Variance vega of option portfolio (red line) with options weighted proportional to $1/K^2$

Log contract

Let us consider the following function:

$$f(F_t) = \frac{2}{T} \left\{ \log \frac{F_0}{F_t} + \frac{F_t}{F_0} - 1 \right\} \tag{4}$$

This function is twice differentiable with derivatives:

$$f'(F_t) = \frac{2}{T} \left(\frac{1}{F_0} - \frac{1}{F_t} \right)$$

and

$$f''(F_t) = \frac{2}{TF_t^2}$$

at time $t = 0$ the function $f(F_t)$ has a value of zero.

To find the dynamic of $f(F_t)$ use Itô's lemma. In general for every smooth twice differentiable function $f(F_t)$ Itô's lemma gives:

$$f(F_t) = f(F_0) + \int_0^T f'(F_t)dF_t + \frac{1}{2}\int_0^T F_t^2 f''(F_t)\sigma_t^2 \qquad (5)$$

Substituting the above introduced function gives obtain expression for the realized variance:

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\left(\log\frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right) - \qquad (6)$$

$$-\frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right) dF_t$$

- Equation (7) shows that the value of a realized variance for the time interval from 0 to T equals to:

$$\frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right) dF_t$$

- continuously rebalanced futures position. This position costs nothing to initiate and easy to replicate;

$$\frac{2}{T}\left(\log \frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right)$$

- **log contract**, static position of a contract that pays $f(F_T)$ at expiry and has to be replicated.

Carr and Madan (2002) argue that the market structure assumed above allows to represent any twice differentiable payoff function $f(F_T)$:

$$f(F_T) = f(k) + f'(k)\big([(F_T - k)^+ - (k - F_T)^+] \qquad (7)$$

$$+ \int_0^k f''(K)(K - F_T)^+ dK$$

$$+ \int_k^\infty f''(K)(F_T - K)^+ dK$$

For $f(F_T)$ expansion around F_0 we gives:

$$\log\left(\frac{F_0}{F_T}\right) + \frac{F_T}{F_0} - 1 = \tag{8}$$

$$= \int_0^{F_0} \frac{1}{K^2}(K - F_T)^+ dK + \int_{F_0}^{\infty} \frac{1}{K^2}(F_T - K)^+ dK$$

To obtain the strike K_{var}^2 of a variance swap take a risk-neutral expectation:

$$K_{var}^2 = \frac{2}{T} e^{rT} \int_0^{F_0} \frac{1}{K^2} P_0(K) dK + \frac{2}{T} e^{rT} \int_{F_0}^{\infty} \frac{1}{K^2} C_0(K) dK \quad (9)$$

But it is impossible to find vanilla options with a complete strike range (from 0 to ∞) traded on the market. How to replicate a fair future realized variance in reality?

Discrete approximation

Demeterfi et al. (1998) proposed approach for finding weights for replicating portfolio of options.
For example, for the call option with strike K_0 the slope of the segment would be:

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0} \tag{10}$$

The second segment - combination of calls with strikes K_0 and K_{1c}:

$$w_c(K_{1c}) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0) \tag{11}$$

Discrete approximation

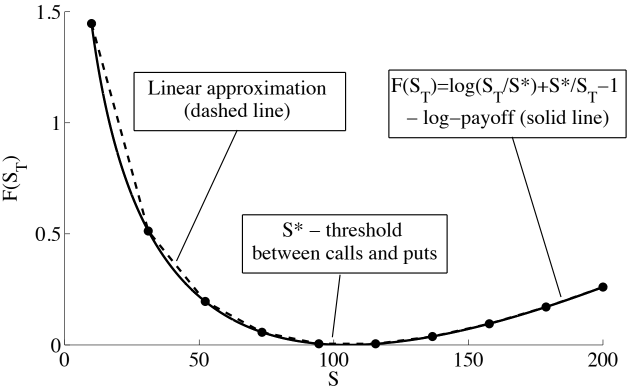


Figure 7: Discrete approximation of a log payoff

Payoff of 1M variance swap

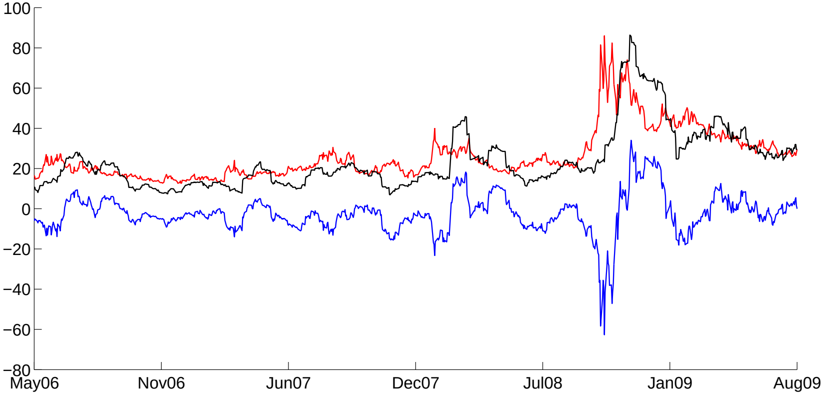


Figure 8: Strike of 1M variance swap, realized 1M volatility, payoff of 1M variance swap (20060217-20091230)

Payoff of 1M variance swap

	Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
2006	-13.84	9.40	-1.89	-2.30	4.75	0.05	2.56
2007	-15.59	5.08	-4.56	-3.95	4.45	-0.33	2.55
2008	-62.66	34.02	-2.49	-2.50	15.12	-0.65	4.70
2009	-18.07	26.01	-1.80	-2.15	7.30	0.83	5.04
All	-62.66	34.02	-2.71	-2.81	9.19	-0.58	9.52

Table 1: Summary statistics of 1M variance swap's payoff for the period from February 2006 to December 2009, annual subsamples and entire data set.

Generalized variance swaps

Modify the floating leg of a standard variance swap (1) with a weight process w_t to obtain:

$$\sigma_R^2 = \frac{252}{T} \sum_{t=1}^{T} w_t \left(\log \frac{F_t}{F_{t-1}} \right)^2 \qquad (12)$$

Corridor and conditional variance swaps

$w_t = w(F_t) = \mathbf{I}_{F_t \in C}$ defines a corridor variance swap with corridor C.

- for $C = [A, B]$ the payoff function is defined by

$$f(F_t) = \frac{2}{T}\left(\log\frac{F_0}{F_t} + \frac{F_t}{F_0} - 1\right)\mathbf{I}_{F_t \in [A,B]} \tag{13}$$

  where $\mathbf{I}$ is the indicator function.

- $C = [0, B]$ gives downward variance swap
- $C = [A, \infty]$ gives upward variance swap

Gamma swaps

$w_t = w(F_t) = F_t/F_0$ defines a price-weighted variance swap or gamma swap with realised variance paid at expiry:

$$\sigma_{gamma} = \sqrt{\frac{252}{T}\sum_{t=1}^{T}\frac{F_t}{F_0}\left(\log\frac{S_t}{S_{t-1}}\right)^2} \cdot 100 \tag{14}$$

The payoff function:

$$f(F_t) = \frac{2}{T}\left(\frac{F_t}{F_0}\log\frac{F_t}{F_0} - \frac{F_t}{F_0} + 1\right) \tag{15}$$

Basket volatility

Basket variance

$$\sigma^2_{Basket} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

replace $\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$,

then $\rho = \dfrac{\sigma^2_{Basket} - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j}$ is the average basket correlation.

Payoff of dispersion strategy

The payoff of the direct dispersion strategy is a sum of variance swap payoffs of each of i-th constituent

$$(\sigma^2_{R,i} - K^2_{var,i}) \cdot N_i$$

and of short position in index swap

$$(K^2_{var,index} - \sigma^2_{R,index}) \cdot N_{index}$$

where

$$N_i = N_{index} \cdot w_i$$

Payoff of dispersion strategy

The payoff of the overall strategy is:

$$N_{index} \cdot \left(\sum_{i=1}^{n} w_i \sigma_{R,i}^2 - \sigma_{R,Index}^2 \right) - ResidualStrike \qquad (16)$$

$$ResidualStrike = N_{index} \cdot \left(\sum_{i=1}^{n} w_i K_{var,i}^2 - K_{var,Index}^2 \right)$$

Conclusions

- Volatility can be traded as an asset
- Future realized volatility can be replicated with option portfolios
- With linear interpolation replication performs well
- The success of the volatility dispersion strategy lies in determining:
  - Direction of the strategy (GARCH volatility forecasts)
  - Constituents for the offsetting variance basket (PCA, DSFM)
  - Proper weights of the constituents (vega-flat strategy, gamma-flat strategy, theta-flat strategy)

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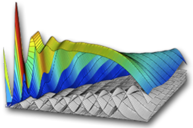

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Bibliography

Bossu, S., Strasser, E. and Guichard, R.
Just what you need to know about Variance Swaps
Equity Derivatives Investor Marketing, Quantitative Research and Development, JPMorgan - London, (May 2005)

Canina, L. and Figlewski, S.
The Informational Content of Implied Volatility
The Review of Financial Studies, 6 (3), 659-681 (1993)

Carr, P. and Lee, R.
Realized Volatility and Variance: Options via Swaps
RISK, 20 (5), 76-83 (2007)

Bibliography

Carr, P. and Lee, R.
Robust Replication of Volatility Derivatives
PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting, (14 April 2008)

Carr, P. and Wu, L.
A Tale of Two Indices
The Journal of Derivatives, 13-29 (Spring, 2006)

Carr, P. and Madan D.
Towards a theory of volatility trading
Volatility, pages 417-427, (2002)

Bibliography

Chriss, N. and Moroko, W.
Market Risk for Volatility and Variance Swaps
RISK, (July 1999)

Demeterfi, K., Derman, E., Kamal, M. and Zou, J.
More Than You Ever Wanted To Know About Volatility Swaps
Goldman Sachs Quantitative Strategies Research Notes, (1999)

Franke, J., Härdle, W. and Hafner, C. M.
Statistics of Financial Markets: An Introduction (Second ed.)
Springer Berlin Heidelberg (2008)

Bibliography

Hull, J.
Options, Futures, and Other Derivatives (7th revised ed.)
Prentice Hall International (2008)

Neil, C. and Morokoff, W.
Realised volatility and variance: options via swaps
RISK, (May, 2007)

Sulima, C. L.
Volatility and Variance Swaps
Capital Market News, Federal Reserve Bank of Chicago,
(March 2001)