Implied Volatility String Dynamics

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Aims

Model and estimate implied volatility surfaces (IVS) for

- trading
- hedging of derivative positions
- risk management.

In these contexts the IVS acts as a very **high-dimensional state variable**.
Practice requires a **low-dimensional representation** of the IVS.

Challenges

- Large number of observations ($>$ 2 million contracts, $>$ 5 000 observations per day).
- Data appear in ‘strings’.
- Strings are not locally fixed, but ‘move’ through the observation space (expiry effect).
- In the moneyness dimension observations may be missing in certain sub-regions for some dates i.

Degenerated Design

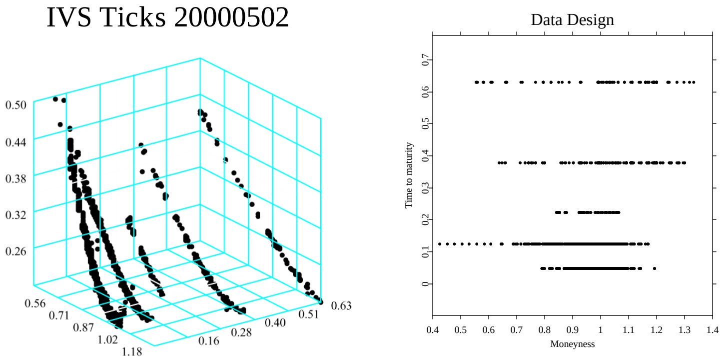


Figure 1: Left panel: call and put implied volatilities observed on 20000502. Right panel: data design on 20000502; ODAX, difference-dividend correction according to Hafner and Wallmeier (2001) applied.

Purpose

A **modelling strategy** in terms of a **dynamic semiparametric factor model (DSFM)** for the (log)-IVS
$Y_{i,j}$ $(i = \text{day}, j = \text{intraday})$:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) \quad . \tag{1}$$

Here $m_l(X_{i,j})$ are smooth factor functions and $\beta_{i,l}$ is a multivariate (loading) time-series.

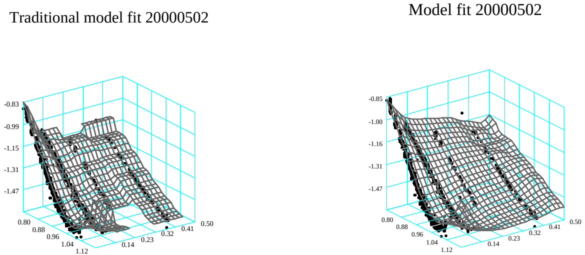


Figure 2: *Traditional model (Nadaraya-Watson estimator) and semi-parametric factor model fit for 20000502. Bandwidths for both estimates* $h_1 = 0.03$ *for the moneyness and* $h_2 = 0.08$ *for the time to maturity dimension.*

Overview

1. aims and generic challenges✓
2. implied volatilities
3. short literature review
4. model
5. algorithm
6. results
7. application
8. outlook

Implied volatilities

Black and Scholes (1973) (BS) formula prices European options under the assumption that the asset price S_t follows a geometric Brownian motion with constant drift and constant volatility coefficient σ:

$$C_t^{BS} = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \,,$$

where $d_{1,2} = \frac{\ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$. $\Phi(u)$ is the CDF of the standard normal distribution, r a constant interest rate, $\tau = T - t$ time to maturity, K the strike price.

Implied volatilities

Volatility $\hat{\sigma}$ as *implied* by observed market prices $\tilde{C}_t$:

$$\hat{\sigma} : \quad \tilde{C}_t - C_t^{BS}(S_t, K, \tau, r, \hat{\sigma}) = 0 \, .$$

Unlike assumed in the BS model, $\hat{\sigma}_t(K, \tau)$ exhibits **distinct, time-dependent** functional patterns across K (**smile or smirk**), and a **term-structure** $T - t$: Thus $\hat{\sigma}_t(K, \tau)$ is interpreted as a **random surface**: the implied volatility surface (IVS).

Related work

One strand of literature models IVS ‘slices’ using PCA:

- Alexander (2001) analyzes fixed strike deviations,
- Skiadopoulos et al. (1999) explore the smile in different maturity buckets,
- Avellaneda and Zhu (1997); Fengler et al. (2002) investigate the term structure.

Related work

Recently, a more comprehensive surface perspective is adopted:

- Fengler et al. (2003) propose a simultaneous decomposition of maturity groups in a **common principal components** framework.
- Cont and da Fonseca (2002) employ the **Karhunen und Loève** decomposition.

This literature does not properly cope with the degenerated design. Estimates are necessarily **biased**.

The semiparametric factor model

Consider DSFM for the IVS:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) \quad , \tag{2}$$

$Y_{i,j}$ is log IV,i denotes the trading day ($i = 1, \ldots, I$),
$j = 1, \ldots, J_i$ is an index of the traded options on day i.
$m_l(\cdot)$ for $l = 0, \ldots, L$ are basis functions in covariables $X_{i,j}$,
and β_i are time dependent factors.

For $m_l(\cdot)$, $l = 0, \dots, L$ consider two different set-ups in $X_{i,j}$:

(A) $X_{i,j}$ is a two-dimensional vector containing time to maturity $\tau_{i,j}$ and forward moneyness, $\kappa_{i,j} = \frac{K}{F(t_{i,j})}$, i.e. strike K divided by futures price $F(t_{i,j}) = S_{t_{i,j}} \exp(r_{\tau_{i,j}} \tau_{i,j})$

(B) as in (A) but with one-dimensional $X_{i,j}$ that only contains $\kappa_{i,j}$.

Here, we focus on (A).

Space and time smoothing

Define estimates of $\widehat{m}_l$ and $\widehat{\beta}_{i,l}$ with $\widehat{\beta}_{i,0} \stackrel{\text{def}}{=} 1$, as minimizers of:

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\}^2 K_h(u - X_{i,j})\, du, \qquad (3)$$

where K_h denotes a two dimensional product kernel,
$K_h(u) = k_{h_1}(u_1) \times k_{h_2}(u_2), \quad h = (h_1, h_2)$ with a one-dimensional kernel $k_h(v) = h^{-1} k(h^{-1} v)$.

Replace in (3) $\widehat{m}_l$ by $\widehat{m}_l + \delta g$ and $\widehat{\beta}_{i,l}$ by $\widehat{\beta}_{i,l} + \delta$. Take derivatives wrt δ, $(1 \le l' \le L, 1 \le i \le I)$:

$$\sum_{i=1}^{I} J_i \widehat{\beta}_{i,l'} \widehat{q}_i(u) = \sum_{i=1}^{I} J_i \sum_{l=0}^{L} \widehat{\beta}_{i,l'} \widehat{\beta}_{i,l} \widehat{p}_i(u) \widehat{m}_l(u), \tag{4}$$

$$\int \widehat{q}_i(u) \widehat{m}_{l'}(u)\, du = \sum_{l=0}^{L} \widehat{\beta}_{i,l} \int \widehat{p}_i(u) \widehat{m}_{l'}(u) \widehat{m}_l(u)\, du, \tag{5}$$

$$\widehat{p}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}),$$

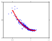
$$\widehat{q}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}) Y_{i,j}.$$

Model characteristics

Consider the case $L = 0$: the log-implied volatilities $Y_{i,j}$ are approximated by a surface $\widehat{m}_0$ not depending on day i. Then,

$$\widehat{m}_0(u) = \frac{\sum_{i,j} K_h(u - X_{i,j}) Y_{i,j}}{\sum_{i,j} K_h(u - X_{i,j})},$$

$\widehat{m}_0$ is equal to the Nadaraya-Watson estimate based on the pooled sample of all days.

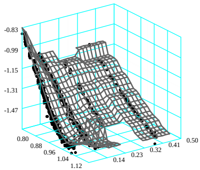


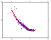
Model characteristics

Consider a fixed day i and $L = 0$:,

$$\widehat{m}_0^{(i)}(u) = \frac{\sum_{j=1}^{J_i} K_h(u - X_{i,j}) Y_{i,j}}{\sum_{j=1}^{J_i} K_h(u - X_{i,j})},$$

Traditional model fit 20000502





Model characteristics

IVS's are fitted in *neighborhoods* of the observed design points $X_{i,j}$, i.e.

- we do not fit the surface on the whole design space on each day (as in a functional PCA (fPCA), Ramsay and Silverman (1997)).
- we circumvent global fits and thus avoid large bias effects caused by the degenerated string design.

Model characteristics

In fPCA factors are eigenfunctions of a covariance operator. Here, the norm:

$$\int f^2(u)\hat{p}_i(u)du \ ,$$

changes each day i, where $\hat{p}_i(u) = J_i^{-1}\sum_{j=1}^{J_i} K_h(u - X_{i,j})$.
Eigenfunctions $\widehat{m}_l$ may not be nested for increasing L:
Hence, the $\widehat{m}_l$ cannot be calculated iteratively, i.e. by moving from $L-1$ components to L components, and so forth.

Model characteristics

In the DSFM framework the IVS's are approximated by surfaces moving in the function space

$$\{\widehat{m}_0 + \sum_{l=1}^{L} \alpha_l \widehat{m}_l : \ \alpha_1, \ldots, \alpha_L \in \mathbb{R}\}.$$

The estimates $\widehat{m}_l$ are not uniquely defined: they can be replaced by estimates that span the same affine space.
Natural choice: orthogonalize $\widehat{m}_l$ in an appropriate function space.
Order the resulting functions according to maximum variance in $\widehat{\beta}_l$.

Orthogonalization

Replace:

$$\widehat{m}_0 \quad \text{by} \quad \widehat{m}_0^{new} = \widehat{m}_0 - \gamma^\top \Gamma^{-1} \widehat{m}$$

$$\widehat{m} \quad \text{by} \quad \widehat{m}^{new} = \Gamma^{-1/2} \widehat{m}$$

$$\widehat{\beta}_i \quad \text{by} \quad \widehat{\beta}_i^{new} = \Gamma^{1/2} (\widehat{\beta}_i + \Gamma^{-1} \gamma)$$

where:

$\widehat{m} = (\widehat{m}_1, ..., \widehat{m}_L)^\top$, $\widehat{\beta}_i = (\widehat{\beta}_{i,1}, ..., \widehat{\beta}_{i,L})^\top$, $\widehat{p}(u) = \frac{1}{I} \sum_{i=1}^{I} \widehat{p}_i(u)$

Γ is (LxL) matrix with $\Gamma_{l,l'} = \int \widehat{m}_l(u) \widehat{m}_{l'}(u) \widehat{p}(u) du$

γ is (Lx1) vector with $\gamma_l = \int \widehat{m}_0(u) \widehat{m}_l(u) \widehat{p}(u) du$.

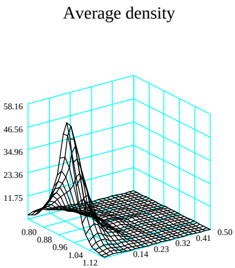


Figure 3: *The average density* $\widehat{p}(u)$

Ordering

Define matrix B with $B_{l,l'} = \sum_{i=1}^{I} \widehat{\beta}_{i,l}\widehat{\beta}_{i,l'}$ and $Z = (z_1, ..., z_L)$ where z_1,...,z_L eigenvectors of B.

Replace:

$$\widehat{m} \quad \text{by} \quad \widehat{m}^{new} = Z^\top \widehat{m}$$

$$\widehat{\beta}_i \quad \text{by} \quad \widehat{\beta}_i^{new} = Z^\top \widehat{\beta}_i$$

The orthonormal basis $\widehat{m}_1, ..., \widehat{m}_L$ is chosen such that $\sum_{i=1}^{I} \widehat{\beta}_{i,1}^2$ is maximal and given $\widehat{\beta}_{i,1}, \widehat{m}_0, \widehat{m}_1$ the quantitiy $\sum_{i=1}^{I} \widehat{\beta}_{i,2}^2$ is maximal and so forth.

Algorithm

The algorithm exploits equations (4) and (5) iteratively:

1. for an appropriate initialization of
   $\beta^{(0)}_{l,i}, \quad i = 1, \ldots, I, \quad l = 1, \ldots, L$
   get an initial estimate of $\widehat{m}^{(0)} = (\widehat{m}_0, ..., \widehat{m}_L)^\top$
2. update $\beta^{(1)}_i, \quad i = 1, \ldots, I,$
3. estimate $\widehat{m}^{(1)}$.
4. go to step 2.

until minor changes occur during the cycle.
Optimization implemented in XploRe, `DSFM.xpl`, Härdle et al. (2000).

Data Overview

	Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
T. to mat.	0.028	2.002	0.142	0.086	0.166	3.658	21.449
Moneyness.	0.287	3.367	0.996	0.997	0.114	0.686	12.026
IV	0.040	0.799	0.297	0.265	0.105	1.289	4.489

Table 1: *Summary statistics from 199901 to 200302. Source: EUREX, ODAX, stored in the SFB 649 FEDC.*

$J_i \approx 5\,200$ observations per day
total time series has $I \approx 1000$ days.
$N = IJ_i \approx 2.8$ million contracts,

Model selection

For a data-driven choice of bandwidths we propose a weighted AIC since the distribution of observations is very unequal:

$$\Xi_{AIC_1} = \frac{1}{N}\sum_{i,j}\{Y_{i,j} - \sum_{l=0}^{L}\widehat{\beta}_{i,l}\widehat{m}_l(X_{i,j})\}^2 w(X_{i,j}) \exp\{\frac{2L}{N}K_h(0)\int w(u)du\},$$

alternatively (computationally easier):

$$\Xi_{AIC_2} = \frac{1}{N}\sum_{i,j}\{Y_{i,j} - \sum_{l=0}^{L}\widehat{\beta}_{i,l}\widehat{m}_l(X_{i,j})\}^2 \exp\{\frac{2L}{N}K_h(0)\frac{\int w(u)du}{\int w(u)p(u)du}\}.$$

w is a given weight function. Putting $w(u) = 1$ delivers common AIC, putting $w(u) = \frac{1}{p(u)}$ give equal weight everywhere.

Model selection

For the model size (L) selection use the:

$$RV(L) = \frac{\sum_i^I \sum_j^{J_i} \{Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2}{\sum_i^I \sum_j^{J_i} (Y_{i,j} - \bar{Y})^2}$$

where $\bar{Y}$ denotes the overall mean of the observations.

L	1-RV(L)	ΔRV
1	0.9638	
2	0.9739	0.0101
3	0.9822	0.0083
4	0.9830	0.0007

Table 2: *Explained variance for the model size.*

Estimation Results

We fit the model for $L = 3$, i.e. there are

- one invariant basis function $\widehat{m}_0$ and
- 3 'dynamic' basis functions $\widehat{m}_1, \widehat{m}_2, \widehat{m}_3$
- 3 time series of $\{\beta_{l,i}\}_{i=1}^{I}$ with $l = 1, 2, 3$

The bandwidths were chosen according to $AIC2$ criterion:
$h_1 = 0.03, h_2 = 0.02$

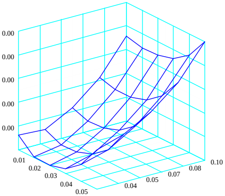


Figure 4: Ξ_{AIC_2} *dependence on the bandwidths.*

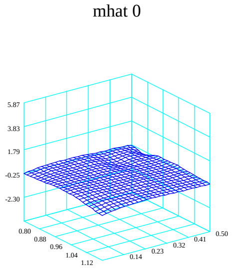
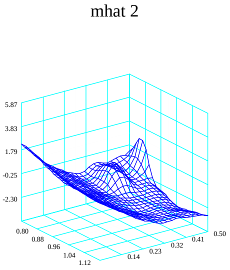


Figure 5: *Invariant basis function* $\widehat{m}_0$ *and dynamic basis function* $\widehat{m}_1$



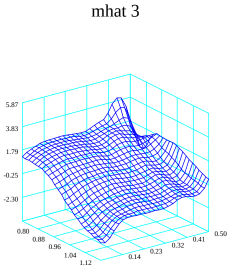


Figure 6: *Dynamic basis functions* $\widehat{m}_2$ *and* $\widehat{m}_3$

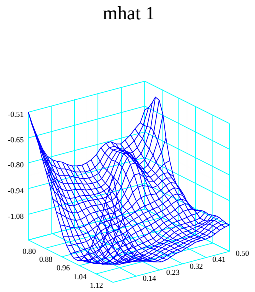


Figure 7: *The dynamic basis function* $\widehat{m}_1$

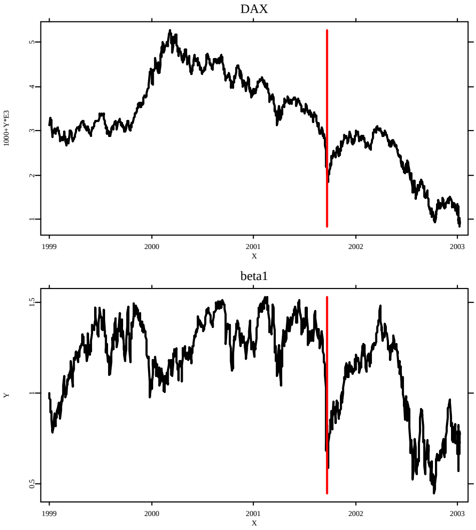


Figure 8: *DAX and time series of weights* $\widehat{\beta}_1$

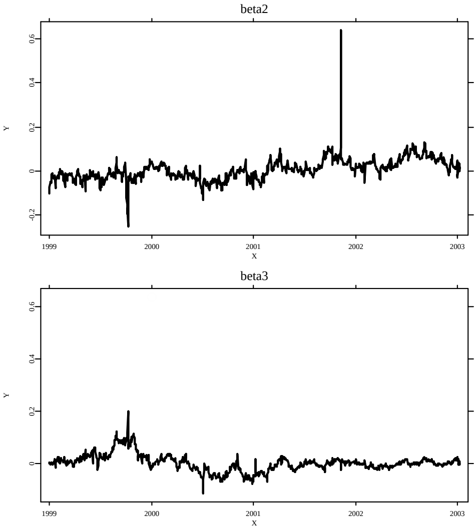


Figure 9: *Time series of weights $\widehat{\beta}_2$ and $\widehat{\beta}_3$*

Correlogram for $\widehat{\beta}_1$, $\widehat{\beta}_2$ and $\widehat{\beta}_3$

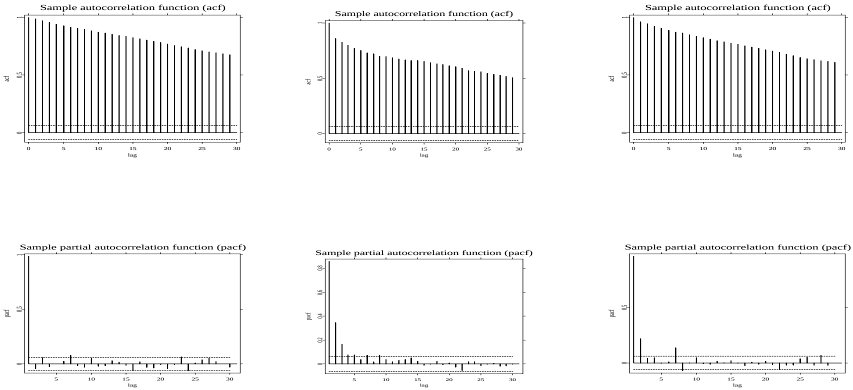


Figure 10: *acf and pacf of* $\widehat{\beta}_1$, $\widehat{\beta}_2$ *and* $\widehat{\beta}_3$ *respectively*

Testing for random walk

coeff.	lag differences	suggested break date	test-value
$\widehat{\beta}_1$	7	2001.11.09	-1.33
$\widehat{\beta}_2$	2	2001.11.09	-1.09
	3		-1.04
$\widehat{\beta}_3$	2	1999.06.08	-3.42*

Table 3: *Unitroot test in the presence of structural break. Critical values for rejecting the hypothesis of unit root are -2.88 at* 5% *significance level and -3.48 at* 1% *significance level. (*) indicate significance at* 5% *level. Lane et al. (2002)*

We model first differences of $\widehat{\beta}_1$, $\widehat{\beta}_2$ and level $\widehat{\beta}_3$ in the form

$$Y_t = (\Delta\widehat{\beta}_1, \Delta\widehat{\beta}_2, \hat{\beta}_3)^\top$$

$$Y_t = \upsilon + A_1 Y_{t-1} + A_2 Y_{t-2} +, \ldots, + A_p Y_{t-p} + \varepsilon_t$$

$Y_t = (Y_{1t}, \ldots, Y_{kt})^\top$ are vectors of the $k = 3$ endogenous variables

$\upsilon = (\upsilon_1, \ldots, \upsilon_k)^\top$ is a vector of intercept terms, A_i are $(K \times K)$ coefficient matrices

ε_t is a white noise with covariance matrix $\Sigma_\varepsilon > 0$

Order Selection Criteria

Lag	ln(FPE)	AIC	SC	HQ
0	-24.34	-15.83	-15.81	-15.82
1	-24.61	-16.10	-16.04	-16.07
2	-24.66	-16.15	-16.05*	-16.11
3	-24.68	-16.17	-16.03	-16.11*
4	-24.68	-16.17	-15.98	-16.10
5	-24.69	-16.16	-15.94	-16.08
6	-24.70*	-16.18*	-15.91	-16.08
7	-24.69	-16.18	-15.87	-16.06
8	-24.69	-16.17	-15.82	-16.04

Table 4: *VAR Lag Order Selection. * indicates lag order selected by the criterion up to a maximum order 8. We chose to apply a VAR(2) as indicated by the SC criterion.*

$$\begin{bmatrix} \Delta\widehat{\beta}_{1t} \\ \Delta\widehat{\beta}_{2t} \\ \widehat{\beta}_{3t} \end{bmatrix} = \begin{bmatrix} 0.12 & 0.22 & -0.09 \\ -0.09 & -0.57 & 0.08 \\ 0.01 & 0.03 & 0.74 \end{bmatrix} \begin{bmatrix} \Delta\widehat{\beta}_{1t,t-1} \\ \Delta\widehat{\beta}_{2t,t-1} \\ \widehat{\beta}_{3t,t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} -0.07 & 0.03 & 0.09 \\ -0.01 & -0.24 & -0.07 \\ -0.01 & 0.01 & 0.23 \end{bmatrix} \begin{bmatrix} \Delta\widehat{\beta}_{1t,t-2} \\ \Delta\widehat{\beta}_{2t,t-2} \\ \widehat{\beta}_{3t,t-2} \end{bmatrix}$$

$$+ \begin{bmatrix} \hat{u}_{1,t} \\ \hat{u}_{2,t} \\ \hat{u}_{3,t} \end{bmatrix}$$

VAR model for first difference levels, $(\Delta\widehat{\beta}_1, \Delta\widehat{\beta}_2, \hat{\beta}_3)^\top$

Model Stability

Time invariance of the model has been evaluated through the roots of the characteristic polynomial for the VAR(2) model as well as coefficient stability through the cumulative sum of squares of the residuals.

roots	modulus
0.97	0.97
$-0.27 \pm 0.4i$	0.48
$0.04 \pm 0.2i$	0.27
-0.23	0.23

Table 5: *Roots of characteristic polynomial for the VAR(2): stability condition is satisied since no root lies outside the unit circle.*

Model Stability

CUSUM-square statistic:

$$S_t = \frac{\sum_{r=k+1}^{t} W_r^2}{\sum_{r=k+1}^{T} W_r^2}$$

W_r^2 (recursive residuals) is the square one-period ahead prediction error. $r = k+1, \ldots, T$ (k, the number of regressors including a constant and T, sample size.
We plot S_r together with significance level lines $E[S_r] \pm C_0$, the statistical "boundaries". C_0 depends on $T - k$ and the significance level desired, see Harvey (1990).

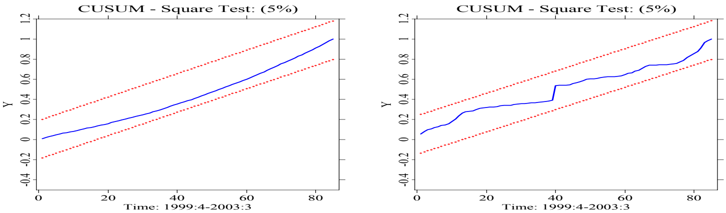


Figure 11: *CUSUM-square statistics for* Δz_1 *and* Δz_2 *equation*

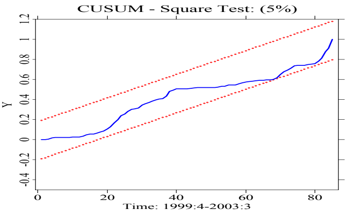


Figure 12: *CUSUM-square statistics for z_3 equation*

Coefficient stability is not rejected as all plots lies within the critical boundaries.

Hedging exotic options

Knock-out options are financial options that become worthless as soon as the underlying reaches a prespecified barrier.

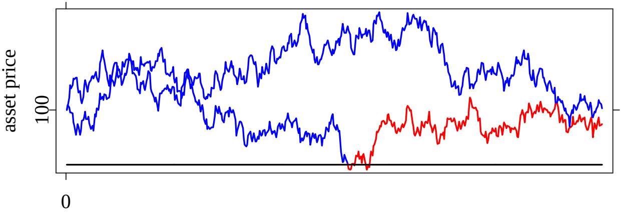


Figure 13: Example of two possible paths of asset's price. When the price hits the barrier (red) the option is no longer valid regardless further evolution of the price.

PROTECT-AKTIENANLEIHEN

DIE PROTECT-AKTIENANLEIHE Kupon p. a.	Basispreis in Euro	PROTECT-Preis in Euro	Alternativ-rückzahlung in Aktien	WKN	Verkaufs-kurs in %
7,5 % Deutsche Bank	60,98	49,00	82	SAL 22W	100,40
5,0 % Deutsche Telekom	15,24	12,75	328	SAL 22Y	100,20
8,0 % SAP	135,14	105,00	37	SAL 234	100,70
10,0 % TUI	16,23	12,50	308	SAL 235	100,00

DAS PRINZIP
Beispiel 7,5 % PROTECT-Aktienanleihe auf Deutsche Bank: Die Anleihe wird am 30. November 2005 zu 100 % zurückgezahlt, sofern die Aktie der Deutsche Bank AG im Xetra-Handelssystem bis zum 23. November 2005 nicht einmal bei bzw. unter dem PROTECT-Preis von 49,00 Euro notiert oder am 23. November 2005 über dem Basispreis von 60,98 Euro schließt. Andernfalls ist die Emittentin berechtigt, als Alternative 82 Aktien je 5.000 Euro Nominalbetrag zu liefern. Die Zinsen in Höhe von 7,5 % werden garantiert gezahlt.

Fälligkeit: *30. November 2005.* ***Anlagebetrag:*** *Nominal* 5.000 Euro oder ein Vielfaches. **Zinszahlung:** Ab 9. November 2004. **Börsenhandel:** Düsseldorf, Frankfurt, Stuttgart. Allein maßgeblich ist der Verkaufsprospekt, dem Sie auch nähere Informationen zu den Chancen und Risiken des Produktes entnehmen können. Das Anlage-Barometer stellt keine Anlageempfehlung dar und ersetzt nicht die individuelle Beratung durch Ihre Hausbank. Den Verkaufsprospekt erhalten Sie kostenlos bei der Emittentin, Sal. Oppenheim jr. & Cie. KGaA, Untermainanlage 1, 60329 Frankfurt am Main. Die Verkaufskurse werden fortlaufend an die Marktentwicklung angepaßt; Stand: 5. November 2004
Service-Telefon 0 69/71 34-22 33____**E-Mail** retailproducts@oppenheim.de____**Internet** www.oppenheim-derivate.de____***Teletext*** *n-tv Tafel 819*

Figure 14: Newspaper advertisement of Sal. Oppenheim's knock-out options (source: Frankfurter Allgemeine Zeitung, November 2004)

Basiswert: DAX 4.130,81 +41,68 +1,02% 11.11.2004 Java-Applet: aktiv Neu Starten

WKN	Typ	Bid	Zeit	Ask	Zeit	Strike	StopLoss	Währung	BV	Fälligkeit
SAL60F	Long	2.470	7:05:14 PM	2.490	7:05:14 PM	3.900,00	3.900,00	XXP	0,01	23.12.2004
SAL6UC	Long	2.650	7:05:24 PM	2.670	7:05:24 PM	3.900,00	3.900,00	XXP	0,01	24.03.2005
SAL60G	Long	2.240	7:05:14 PM	2.260	7:05:14 PM	3.925,00	3.925,00	XXP	0,01	23.12.2004
SAL6U9	Long	1.970	7:05:14 PM	1.990	7:05:14 PM	3.950,00	3.950,00	XXP	0,01	23.12.2004
SAL6UA	Long	1.730	7:05:14 PM	1.750	7:05:14 PM	3.975,00	3.975,00	XXP	0,01	23.12.2004
SAL6UB	Long	1.470	7:05:14 PM	1.490	7:05:14 PM	4.000,00	4.000,00	XXP	0,01	23.12.2004
SAL4VM	Short	0.160	7:05:14 PM	0.180	7:05:14 PM	4.150,00	4.150,00	XXP	0,01	23.12.2004
SAL4VN	Short	0.410	7:05:14 PM	0.430	7:05:14 PM	4.175,00	4.175,00	XXP	0,01	23.12.2004
SAL1S6	Short	0.660	7:05:24 PM	0.670	7:05:24 PM	4.200,00	4.200,00	XXP	0,01	23.12.2004
SAL2GN	Short	0.610	7:05:27 PM	0.630	7:05:27 PM	4.200,00	4.200,00	XXP	0,01	24.03.2005

Figure 15: Bid-/Ask information of Sal. Oppenheim's knock-out options

Hedging exotic options

In BS world prices of barrier options are given analytically, all greeks can be calculated directly.
There exists static replication for some barrier option if:

- the underlying has no drift
- the IV on the market only depends on time not on strike

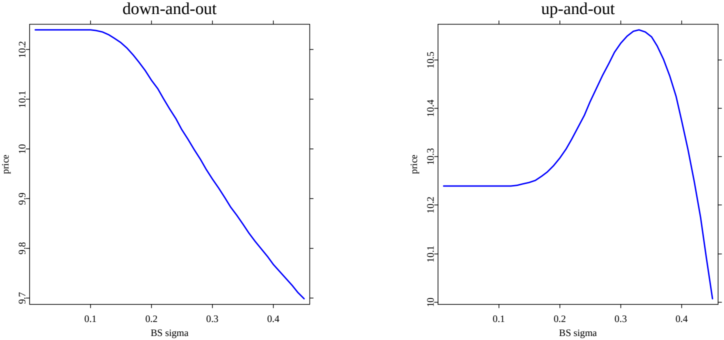


Figure 16: *Price of the call knock-out barrier options as a function of BS-σ. Asset value* $S_0 = 90$, *strike price* $K = 80$ *time to maturity* $\tau = 0.1$ *interest rate* $r = 0.03$. *Left panel: barrier* $B = 80$. *Right panel: barrier* $B = 120$.

Example

Consider a short position in a knock-out call option (C^{KO}) with strike 100 and barrier 90. Consider also one long position in a European call with strike 100 and a short position in 100/90 European puts with strike 81.

- if spot is at the barrier level 90 call and put would be worth the same
- if barrier was not reached before maturity the payoff of C^{KO} is equal to the payoff of the call

C^{KO} is replicated with vanilla options.

Position	Value at time t hits barrier	Value at time T doesn't hit barrier
C	$BS_{call}(K = 100)$	$(S_T - 100)^+$
$-100/90P$	$-\frac{100}{90} BS_{put}(K = 81)$	0
$-C^{KO}$	0	$-(S_T - 100)^+$
Sum	0	0

For each time t and each value of σ if $r = 0$ and $S_t = 90$ then $BS_{call}(K = 100) = \frac{100}{90} BS_{put}(K = 81)$

Dynamic hedging

Use approximation of the option value changes and adjust constantly the hedge portfolio.

$$\Delta C^{KO}(\Delta S, \Delta\sigma) \approx \frac{\partial C^{KO}}{\partial S}\Delta S + \frac{\partial C^{KO}}{\partial \sigma}\Delta\sigma$$

The changes in the asset price (delta risk) can be hedge the asset itself. The changes in volatility (vega risk) can be hedge with at-the-money plain vanilla call option (C).

Dynamic hedging

The sensitivity of the hedge portfolio $HP = a_1 S + a_2 C$ w.r.t. S and σ should be equal to the sensitivity of the C^{KO}. The hedge coefficients $a1, a2$ are given by the equation:

$$\begin{pmatrix} 1 & \frac{\partial C}{\partial S} \\ 0 & \frac{\partial C}{\partial \sigma} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial S} \\ \frac{\partial C^{KO}}{\partial \sigma} \end{pmatrix}$$

Local Volatility Model

In local volatility (LV) models the asset price dynamics are governed by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S_t, t) dW_t \tag{6}$$

where W_t is a Brownian motion, μ the drift and $\sigma(S_t, t)$ the local volatility function which depends on the asset price and time only.

For pricing the options the partial differential equation (6) is solved. Price depends on the **entire** IVS. From the IVS one can calculate $C_t(K, T)$.
Dupire formula:

$$\sigma^2(S_t, t) = 2\frac{\frac{\partial C_t(K,T)}{\partial T} + rK\frac{\partial C_t(K,T)}{\partial K}}{K^2\frac{\partial^2 C_t(K,T)}{\partial K^2}}$$

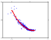
gives the local volatility surface $\sigma(S_t, t)$.

Hedging exotic options

Most greeks can be calculated:

- Delta, $\frac{\partial C^{KO}}{\partial S}$, gamma, $\frac{\partial^2 C^{KO}}{\partial S^2}$ and theta, $\frac{\partial C^{KO}}{\partial t}$, can be read from the grid of the finite difference scheme;
- rho, $\frac{\partial C^{KO}}{\partial r}$, and dividend-rho, $\frac{\partial C^{KO}}{\partial \delta}$, are typically computed via a difference quotient assuming a flat term structure.

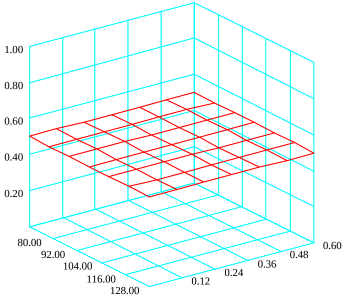
What about the vega ??
The usual vega, $\frac{\partial C^{KO}}{\partial \sigma}$ cannot be used since the entire IVS is input.



Classical vega hedging

Classical vega hedging corresponds to parallel move of IVS

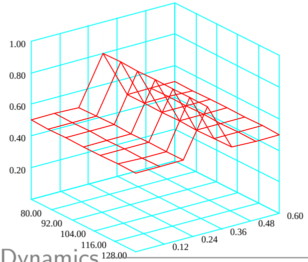
- In BS there is only one volatility number
- In LV it protects only of parallel move of the smile (β_1 effect)



Bucket hedging

With term structure of the IVS one may compute a bucket vega hedging. It provides a sensitivity measure of parallel movements over each maturity string.

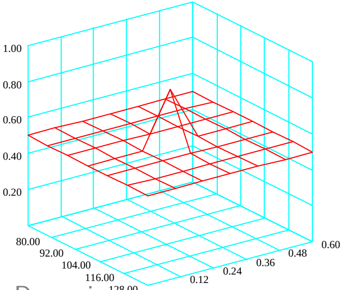
- The procedure indicates which European option maturities should be used for hedging
- Sensitivity related to strike is not given



Superbucket hedging

In superbucket analysis one has to compute sensitivity of exotics w.r.t. a move of each individual implied volatility.

- Sensitivity by strike and maturity is obtained
- The calculation needs to be done for each single point



Vega-hedging of the two DSFM factors

In DSFM the IV decomposition is given only by $L+1$ factors:

$$\widehat{\sigma}_i = \exp\left(\sum_{l=0}^{L} \widehat{\beta}_{i,l}\,\widehat{m}_l\right).$$

We can compute the sensitivities w.r.t. the factor loadings $\widehat{\beta}_l$!
From the interpretations, we receive an immediate understanding of the sensitivities:

- $\frac{\partial}{\partial \widehat{\beta}_1}$ is an **up-and-down shift vega** of the IVS;
- $\frac{\partial}{\partial \widehat{\beta}_3}$ is a **slope shift vega** of the IVS.

How to compute the hedge ratios

Take two hedge portfolios HP_1 and HP_2.
Compute the sensitivities of the hedge portfolios and the knock-out option with respect to $\widehat{\beta}_1$ and $\widehat{\beta}_3$.
Solve

$$\begin{pmatrix} \frac{\partial HP_1}{\partial \widehat{\beta}_1} & \frac{\partial HP_2}{\partial \widehat{\beta}_1} \\ \frac{\partial HP_1}{\partial \widehat{\beta}_3} & \frac{\partial HP_2}{\partial \widehat{\beta}_3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial \widehat{\beta}_1} \\ \frac{\partial C^{KO}}{\partial \widehat{\beta}_3} \end{pmatrix}$$

for the hedge ratios a_1, a_2.

Choice of the hedge portfolio

Idea:
choose HP_1 and HP_2 with *maximum exposure* to $\widehat{\beta}_1$ and $\widehat{\beta}_3$, respectively:

HP_1 should be most sensitive to up-and-down shifts:
use a portfolio of at-the-money plain vanilla options;

HP_2 should be most sensitive to slope changes:
use a portfolio of vega-neutral risk reversals.

Then $\frac{\partial HP_1}{\partial \widehat{\beta}_3} \approx 0$ and $\frac{\partial HP_2}{\partial \widehat{\beta}_1} \approx 0$.

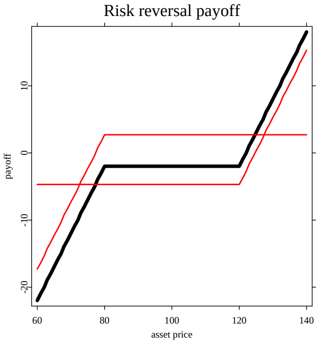


Figure 17: *The payoff of the risk reversal. It is compounded from long call with strike* $K_1 = 120$ *and short put with strike* $K_2 = 80$*.*

Outlook

Agenda:

- local-linear smoothing ✓
- data driven choice of L (number of m), and bandwidth h ✓
- forecasting exercise (almost done)
- investigate obvious relations to **Kalman Filtering**, Fengler et al. (2005):

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \epsilon_i \tag{7}$$

$$\beta_i = \widetilde{\beta}_i(\theta) + \eta_i \tag{8}$$

Outlook

Agenda:

- hedging empirical studies
- estimation of state price density (SPD)

$$f_{T-t}(K) = e^{r(T-t)} \frac{\partial^2 C_t(K,T)}{\partial K^2} \tag{9}$$

where $f_{T-t}(K)$ is SPD of the time T taken in the time t

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