# Network Quantile Autoregression

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# Networks

- High dimensional networks
- Complex risk channels
- Dynamic tail event driven network

Figure 1: TENET movie

# Challenges

- Tails of conditional distribution
- Quantile autoregression
- Herding and impulse effects

# Financial Risk Meter (FRM)

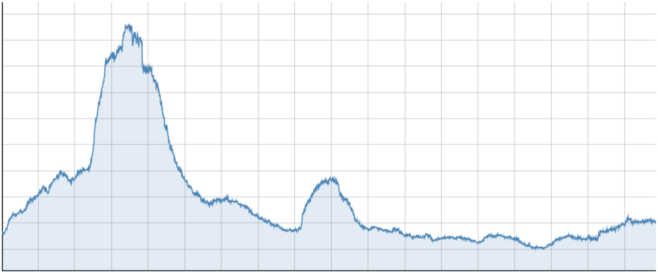


Figure 2: frm.wiwi.hu-berlin.de

# CRyptocurrency IndeX (CRIX)


Figure 3: crix.hu-berlin.de

# Default Intensities in a network topology

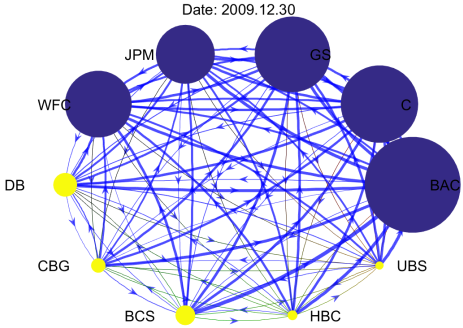



Figure 4: Node size: "TO". Edge thickness: Average edge weight.

# Network dynamics

- NAR method, Zhu et al.(2015)
- Banking, environmental statistics
- Quantile autoregression model, Koenker & Xiao (2006)

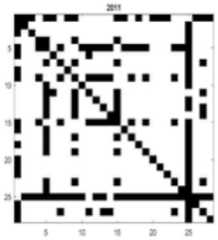



Figure 5: Adjacency matrix of SIFI.

# Challenges

- Model dynamics
- Dimension reduction
- Dynamic tail event methods

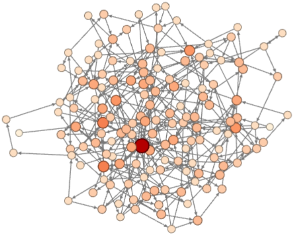


Figure 6: Power-law distribution network.

# Outline

1. Motivation ✓
2. Network quantile autoregression model
3. Simulations
4. Applications
5. Discussion

# CoVaR

- *CoVaR* technique (AB)
- Two linear quantile regressions

$$\begin{aligned} X_{i,t} &= \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t}, \\ X_{j,t} &= \alpha_{j|i} + \beta_{j|i} X_{i,t} + \gamma_{j|i}^\top M_{t-1} + \varepsilon_{j,t}. \end{aligned}$$

$X_{i,t}$ log return, $M_{t-1}$ lagged macro variables.

- $F_{\varepsilon_{i,t}}^{-1}(\tau|M_{t-1}) = 0$ and $F_{\varepsilon_{j,t}}^{-1}(\tau|M_{t-1}, X_{i,t}) = 0$, then

$$\begin{aligned} \widehat{VaR}_{i,t}^{\tau} &= \widehat{\alpha}_i + \widehat{\gamma}_i^\top M_{t-1}, \\ \widehat{CoVaR}_{j|i,t}^{\tau} &= \widehat{\alpha}_{j|i} + \widehat{\beta}_{j|i} \widehat{VaR}_{i,t}^{\tau} + \widehat{\gamma}_{j|i}^\top M_{t-1}. \end{aligned}$$

# Network Model

- $U_{it}$ $(1 \le i \le N, 1 \le t \le T)$ i.i.d. uniform rv's. Nodal covariates $Z_i \in \mathbb{R}^q$

$$\begin{aligned} Y_{it} &= \beta_0(U_{it}) + \sum_{l=1}^{q} Z_{il}\gamma_l(U_{it}) + \beta_1(U_{it}) n_i^{-1} \sum_{j=1}^{N} a_{ij} Y_{i(t-1)} \\ &\quad + \beta_2(U_{it}) Y_{i(t-1)} \stackrel{\text{def}}{=} g_\theta(U_{it}), \end{aligned} \tag{1}$$

$\beta_j$ and $\gamma_l$ monotone functions, $n_i = \sum_{j \ne i} a_{ij}$, $(a_{ij})$ is an adjacency matrix, where $a_{ij} = 1$ if there is an edge from $i$ to $j$, otherwise $a_{ij} = 0$. $Z_{il}$ node-specific variable, $\mathbb{Y}_t = (Y_{1t}, \cdots, Y_{Nt})^\top \in \mathbb{R}^N$

# Quantile Regression

Under assumption $F^{-1}_{\varepsilon_i|X_i}(\tau) = 0$

$$\begin{aligned} Y_i &= \theta^\top X_i + \varepsilon_i, \\ Y_i &= \theta^\top X_i + \beta(U_i) \end{aligned}$$

$U_i \sim U[0,1]$, $\beta$ monotone increasing.

$$Q_{(Y|X)}(\tau) = \theta^\top X_i + \beta(\tau)$$

▸ More details

# A minimum contrast approach

Quantile function of $Y$ given $X = (Z_i, \mathbb{Y}_{t-1})$.

$$
\begin{aligned}
Q_{Y_{it}}(\tau | Z_i, \mathbb{Y}_{t-1}) &= \beta_0(\tau) + \sum_{l=1}^{q} Z_{il}\gamma_l(\tau) + \beta_1(\tau) n_i^{-1} \sum_{j=1}^{N} a_{ij} Y_{j(t-1)} \\
&\quad + \beta_2(\tau) Y_{i(t-1)},
\end{aligned}
$$

- $Y_{j(t-1)}$ impact of the same node.
- $\beta_1(\tau)$ *network function*.
- $\beta_2(\tau)$ *momentum function*.

  $\theta(u) \stackrel{\text{def}}{=} \{\beta_0(u), \beta_1(u), \beta_2(u), \gamma_1(u), \cdots, \gamma_q(u)\}^\top$

# A minimum contrast approach

Estimate $\theta(\tau)$:

$$\widehat{\theta}(\tau) = \arg\min_\theta \sum_i \sum_t \rho_\tau\{Y_{it} - g_\theta(\tau)\}, \tag{2}$$

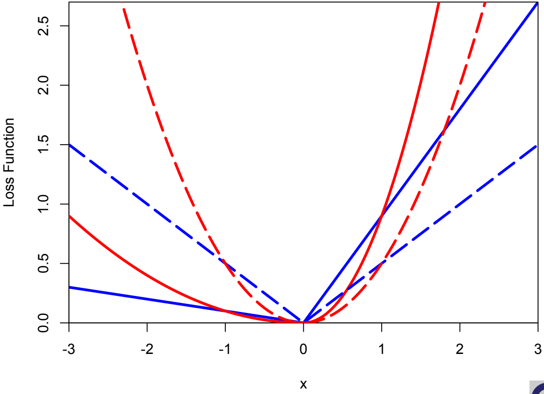
where

$$\rho_\tau(u) = \tau u \mathbf{1}\{u \in (0, \infty)\} - (1-\tau) u \mathbf{1}\{u \in (-\infty, 0]\}. \tag{3}$$

The conditional pdf of $Y_{it}$ may then be estimated:

$$\hat{f}_{Y_{it}|\mathcal{F}_{t-1}}(F_{it}^{-1}(\tau)) = (\tau_i - \tau_{i-1})/\{\hat{Q}_{Y_{it}|\mathbb{Y}_{t-1}}(\tau_i) - \hat{Q}_{Y_{it}|\mathbb{Y}_{t-1}}(\tau_{i-1})\}. \tag{4}$$

# Asymmetric Loss Functions




LQRcheck

Figure 7: Asymmetric Loss Functions for Quantile and Expectile, $\tau = 0.9$: a solid line, $\tau = 0.5$: a dashed line. Quantiles and Expectiles

# NQAR a convenient dynamics

NQAR model (1):

$$\mathbb{Y}_t = \Gamma + G_t \mathbb{Y}_{t-1} + V_t, \quad (5)$$

$\Gamma = \mathsf{E}(\mathbf{B}_{0t}) = c_0 \mathbf{1}_N \in \mathbb{R}^N$, $c_0 = b_0 + c_Z$, $b_0 = \int_0^1 \beta_0(u)du$,
$c_Z = \mathsf{E}(Z_1)^\top r$, $r = \left( \int_0^1 \gamma_l(u)du, 1 \leq l \leq q \right)^\top \in \mathbb{R}^q$.
$\mathbf{B}_{0t} = \{\beta_0(U_{it}) + \sum_l Z_{il}\gamma_l(U_{it}), 1 \leq i \leq N\}^\top$

$$G_t = \mathbf{B}_{1t} W + \mathbf{B}_{2t} \quad (6)$$

$\mathbf{B}_{1t} = \text{diag}\{\beta_1(U_{it}), 1 \leq i \leq N\}, \mathbf{B}_{2t} = \text{diag}\{\beta_2(U_{it}), 1 \leq i \leq N\}^\top$
$W = (n_i^{-1} a_{ij})$ is the row-normalized adjacency matrix.
$V_t = \mathbf{B}_{0t} - \Gamma$ i.i.d. with mean $\mathbf{0}$ and covariance $\Sigma_V \in \mathbb{R}^{N\times N}$.

▸ Definition of $\Sigma_V$

# Definitions

Explanatory variables: $X_{it} \stackrel{\text{def}}{=} (1, Z_i^\top, n_i^{-1} \sum_{j=1}^N a_{ij} Y_{jt}, Y_{it})^\top$.

$\widehat{\Omega}_0 = (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T X_{it} X_{it}^\top$,

$\widehat{\Omega}_1 = (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T f_{it}\big(X_{it}^\top \theta(\tau)\big) X_{it} X_{it}^\top$

for $\tau \in (0,1)$, choose an appropriate sequence of $\tau_l$

$\hat{f}_{it}(F_{it}^{-1}(\tau)) = [X_{i(t-1)}^\top \{\hat{\theta}(\tau_l) - \hat{\theta}(\tau_{l-1})\}]^{-1}(\tau_l - \tau_{l-1})$ at $\tau \in [\tau_{l-1}, \tau_l]$

# Stationarity

## Theorem

*Under assumption C1-C4* Details, $\{\mathbb{Y}_t\}$ *is covariance stationary and there exists a unique solution which has the form of*

$$\mathbb{Y}_t = \sum_{l=0}^{\infty} \Pi_l \Gamma + \sum_{l=0}^{\infty} \Pi_l V_{t-l}, \quad (7)$$

*where* $\Pi_l = \prod_{k=1}^{l} G_{t-k+1}$ *see (6) for* $l \geq 1$ *and* $\Pi_0 = I_N$.

$G_t = \mathbf{B}_{1t} W + \mathbf{B}_{2t}$, $\Gamma = \mathsf{E}(\mathbf{B}_{0t}) = c_0 \mathbf{1}_N \in \mathbb{R}^N$, $V_t = \mathbf{B}_{0t} - \Gamma$.

# Asymptotic Normality

**Theorem**

*Assume* $c_\beta < 1$ *and* $\mathsf{E}(|V_{it}|^4) < M$, *where* $c_\beta = \|\beta_1\|_4 + \|\beta_2\|_4$ *with* $\|\beta_j\|_4 = \mathsf{E}\{\beta_j(U_{it})^4\}^{1/4}$ *(*$j = 1, 2$*). Then:*

$$\sqrt{T}(\overline{\mathbb{Y}}_T - \mu_Y) \xrightarrow{\mathcal{L}} \mathsf{N}(0, \Sigma_Y) \tag{8}$$

*as* $T \to \infty$, *and* $\Sigma_Y = \mathrm{Cov}(\mathbb{Y}_t)$.

# Impulse Analysis

Stimulus $\Delta = (\delta_1, \ldots, \delta_N)^\top \in \mathbb{R}^N$, shocks $V_t = \mathbf{B}_{0t} - \Gamma$.

Creates the *impulse effect* $\mathrm{IE}_{t,t+l} = \prod_{k=0}^{l-1} G_{t+l-k}\Delta$.

- Average IE: $\mathrm{E}(\mathrm{IE}_{t,t+l}) = G^l\Delta = (b_1 W + b_2 I_N)^l \Delta$,
- Interval IE: $\mathrm{IIE}_{l,\tau_1\tau_2} = (c_{\beta_1,\tau_1\tau_2} W + c_{\beta_2,\tau_1\tau_2} I_N)^l \Delta$,
- IE Intensity: $\mathrm{IEI}_{l,\tau} = \left\{\beta_1(\tau) W + \beta_2(\tau) I_N\right\}^l \Delta$.

Definition of $b_1$, $b_2$, $c_{\beta_1,\tau_1\tau_2}$ and $c_{\beta_2,\tau_1\tau_2}$

# An amuse gueule of theory

## Theorem

*Under assumption C1-C4* Details, *we can prove that*

$$\widehat{\theta}(\tau) - \theta(\tau) = (NT)^{-1}\Sigma_\theta(\tau)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} X_{it}\psi_\tau(V_{it\tau}) + r_{NT}(\tau), \quad (9)$$

*where* $\Sigma_\theta(\tau) = \Omega_1^{-1}\Omega_0\Omega_1^{-1}$, Definition of $\Omega_0$ $V_{it\tau} = Y_{it} - g_{\theta,i(t-1)}(\tau)$, $sup_{\tau\in B}|r_{NT}(\tau)| = \mathcal{O}_p\left((NT)^{-1/2}\right)$. *This leads to the consistency result that* $\widehat{\theta}(\tau) \xrightarrow{p} \theta(\tau)$ *as* $\min\{N, T\} \to \infty$ *uniformly for* $\tau \in B$, *where* $B$ *is a compact set in* $(0, 1)$.

# Antipasti Theory

### Theorem

*Under assumption C1-C4* Details, *we have*

$$\sqrt{NT}\Sigma_\theta^{-1/2}(\tau)\{\widehat{\theta}(\tau)-\theta(\tau)\} \xrightarrow{\mathcal{L}} B_p(\tau)$$

### Lemma

*Under assumption C1-C4, we can prove that, for any fixed* $\tau \in B$.

$$\sqrt{NT}\Sigma_\theta^{-1/2}(\tau)\{\widehat{\theta}(\tau)-\theta(\tau)\} \xrightarrow{\mathcal{L}} \mathrm{N}(0,\tau(1-\tau)I_k)$$

*as* $\min\{N, T\} \to \infty$.

# Setup

The baseline, network, and the momentum function are set to be

- $\beta_0(u) = u$,
- $\beta_1(u) = 0.1\Phi(u)$,
- $\beta_2(u) = 0.4\{1 + \exp(u)\}^{-1} \exp(u)$.

# Setup

The dimension of (i.e., $Z_i$) is 5. The nodal functions

- $\gamma_1(u) = 0.5\Phi(u, 1, 2)$,
- $\gamma_2(u) = 0.3\mathbb{G}(u, 2, 2)$,
- $\gamma_3(u) = 0.2\mathbb{G}(u, 2, 2)$,
- $\gamma_4(u) = 0.25\mathbb{G}(u, 3, 2)$,
- $\gamma_5(u) = 0.2\mathbb{G}(u, 2, 1)$.

QNAR

# Setup

$\mathbb{G}(\cdot, a, b)$ is the Gamma distribution function Example of Gamma distribution

with shape parameter $a$ and scale parameter $b$.

$U_{it}$s $(1 \le i \le N, 1 \le t \le T)$ i.i.d. $N(0,1)$ and

$t$-distribution with 5 degrees of freedom.

$Z_i = (Z_{i1}, \cdots, Z_{i5})^\top \in \mathbb{R}^5 \sim N(\mathbf{0}, \Sigma_z)$,

$\Sigma_z = (\sigma_{j_1 j_2})$ and $\sigma_{j_1 j_2} = 0.5^{|\gamma_1 - \gamma_2|}$.

$\mathbb{Y}_0 = (1 - \beta_1^\tau - \beta_2^\tau)^{-1} \beta_0^\tau \mathbf{1}$, $\beta_j^\tau = \phi_j\{F^{-1}(\tau)\}$

and $F(\cdot)$ is the cdf of $U_{it}$. $\tau = \{0.1, 0.2, \cdots, 0.9\}$.

# Network structures

- (Dyad Independence Model) Dyad defined as $D_{ij} = (a_{ij}, a_{ji})$,
  $\mathrm{P}(D_{ij} = (1, 1)) = 20N^{-1}$
  $\mathrm{P}(D_{ij} = (1, 0)) = \mathrm{P}(D_{ij} = (0, 1)) = 0.5N^{-0.8}$.
- (Stochastic Block Model) Randomly assign for each node a block label which is indexed from 1 to $K$. Then set $P(a_{ij} = 1) = 0.3N^{-0.3}$ if $i$ and $j$ are in the same block, and $\mathrm{P}(a_{ij} = 1) = 0.3N^{-1}$.
- (Power-law Distribution Network) $d_i = \sum_j a_{ji}$ discrete power-law distribution $\mathrm{P}(d_i = k) = ck^{-\alpha}$, $c$ normalizing constant and $\alpha$ is set $\alpha = 2.5$

# Visualization of Simulated Networks

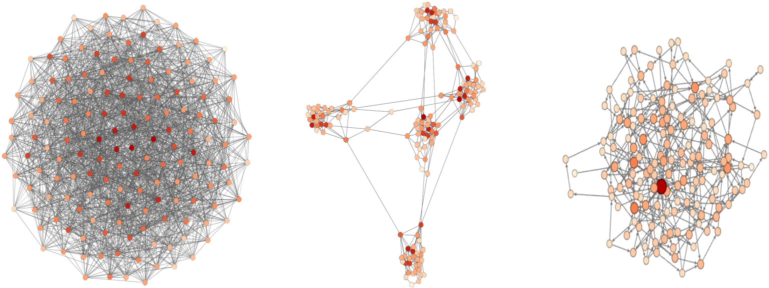


Figure 8: The left panel: dyad independence network; The middle panel: stochastic block model; the right panel: power-law distribution network.

# Estimation of Coefficients

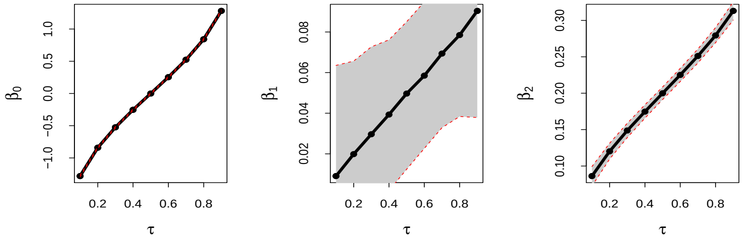



Figure 9: The estimated $\beta_0$ to $\beta_2$ against $\tau$. Dyad independence network.

Table 1: Simulation Results for dyad independence network with 500 Replications. The RMSE ($\times 10^2$) and the Coverage Probability (%) are reported for $\beta_0$ to $\beta_1$. The RMSE is also reported for $\gamma$. Lastly, the network density $\{N(N-1)\}^{-1}\sum_{i_1,i_2} a_{i_1 i_2}$ is computed and given.

| $N$ | Dist. | $\beta_0$ | $\beta_1$ | $\beta_2$ | $\gamma$ | ND |
|---|---|---|---|---|---|---|
| | | | $\tau = 0.1$ | | | |
| 100 | $Z$ | 2.58(95.0) | 6.83(94.8) | 2.50(94.6) | 3.09 | 22.7 |
| | $T$ | 3.52(95.8) | 8.39(95.6) | 2.42(94.8) | 4.22 | |
| 500 | $Z$ | 1.13(96.2) | 3.20(96.2) | 1.08(94.8) | 1.32 | 4.7 |
| | $T$ | 1.42(97.0) | 3.69(94.8) | 1.06(95.2) | 1.85 | |
| 1000 | $Z$ | 0.76(95.6) | 2.44(94.4) | 0.74(95.2) | 0.91 | 2.4 |
| | $T$ | 1.03(97.0) | 2.65(95.0) | 0.77(94.8) | 1.29 | |
| | | | $\tau = 0.9$ | | | |
| 100 | $Z$ | 2.54(94.2) | 6.38(95.8) | 2.36(94.6) | 2.93 | 22.7 |
| | $T$ | 3.81(94.6) | 7.46(94.8) | 2.37(95.0) | 4.21 | |
| 500 | $Z$ | 1.14(95.0) | 3.31(93.4) | 1.05(94.6) | 1.27 | 4.7 |
| | $T$ | 1.46(96.6) | 3.30(96.2) | 1.02(94.2) | 1.81 | |
| 1000 | $Z$ | 0.81(95.6) | 2.17(95.2) | 0.77(93.4) | 0.92 | 2.4 |
| | $T$ | 1.07(95.2) | 2.34(96.0) | 0.79(92.2) | 1.22 | |

# Dataset

- $N = 2442$ stocks from the Chinese A share market.
- Traded in Shanghai Stock Exchange and Shenzhen Stock Exchange in 2013.
- $Y_{it}$ weekly absolute return volatility.

# Firm specific variables

- SIZE (measured by the logarithm of market value),
- BM (book to market ratio),
- PR (increased profit ratio compared to the last year),
- AR (increased asset ratio compared to the last year),
- LEV (leverage ratio),
- Cash (cash flow of the firm).

# Descriptive statistics of financial network

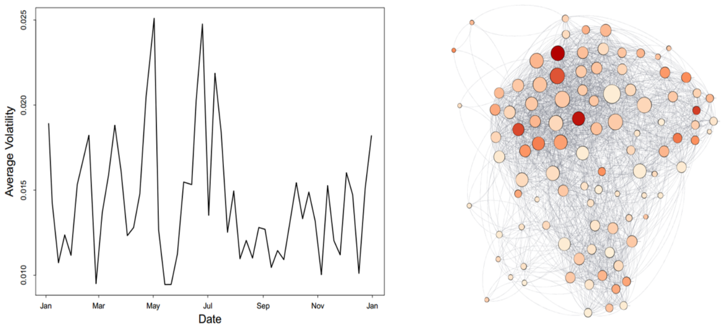



Figure 10: The left panel: the average stock volatility of Chinese A stock market in 2013; the right panel: the common shareholder network of top 100 market value stocks in 2013. The larger and darker points imply higher market capitalization.

# The influential power

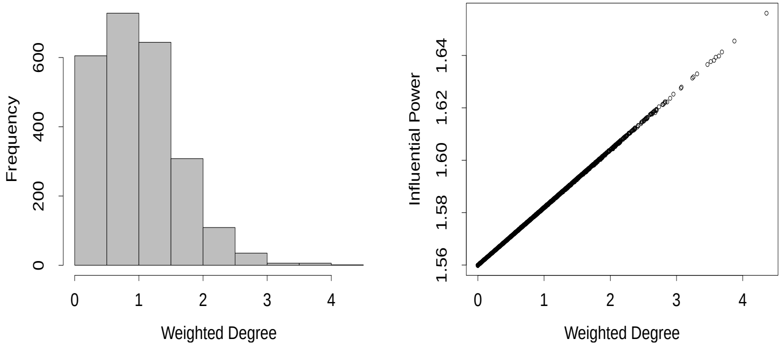



Figure 11: The left panel: the histogram of the weighted degrees; the right panel: the influential power against weighted degrees.

# Impulse analysis

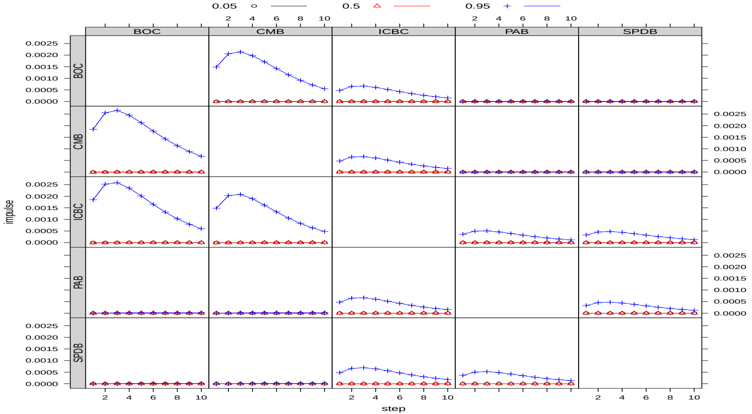



Figure 12: Impulse analysis for $\tau = 0.05, 0.5, 0.95$. The cross-sectional impulse effect intensity between BOC, CMB, ICBC, PAB, and SPDB are given. The impulse direction is from column to row.

Table 2: The detailed NQAR analysis results for the Stock dataset ($\tau$ =0.05, 0.5, 0.95). The yearly estimates ($\times 10^{-2}$) are reported with the standard error ($\times 10^{-2}$) given in parentheses. The p-values are also reported.

| | $\tau = 0.05$ | | $\tau = 0.5$ | | $\tau = 0.95$ | |
|---|---|---|---|---|---|---|
| | Est. | *p*-value | Est. | *p*-value | Est. | *p*-value |
| $\hat{\beta}_0$ | 0.05 (0.00) | $< 0.01$ | 1.00 (0.04) | $< 0.01$ | 2.96 (0.13) | $< 0.01$ |
| $\hat{\beta}_1$ | 0.00 (0.02) | 0.99 | -0.04 (0.77) | 0.95 | 6.09 (2.16) | $< 0.01$ |
| $\hat{\beta}_2$ | 4.16 (0.14) | $< 0.01$ | 35.70 (0.47) | $< 0.01$ | 67.84 (1.13) | $< 0.01$ |
| SIZE | 0.00 (0.01) | 0.98 | -1.00 (0.09) | $< 0.01$ | -4.10 (0.28) | $< 0.01$ |
| BM | 0.00 (0.01) | 0.99 | -0.29 (0.04) | $< 0.01$ | -0.71 (0.25) | $< 0.01$ |
| PR | 0.00 (0.00) | 1.00 | -0.30 (0.12) | 0.01 | 0.39 (0.38) | 0.31 |
| AR | -0.02 (0.03) | 0.53 | -0.66 (0.11) | $< 0.01$ | -0.47 (0.36) | 0.20 |
| CASH | -0.01 (0.01) | 0.03 | -0.14 (0.06) | 0.01 | -0.05 (0.27) | 0.86 |
| LEV | 0.00 (0.01) | 0.97 | -0.79 (0.05) | $< 0.01$ | -2.42 (0.44) | $< 0.01$ |

- Backtesting
- Incorporating shadow banking sectors
- ...

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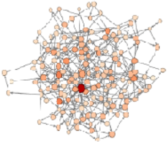

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# Assumptions

(C1) (Moment Assumption) Assume $c_\beta < 1$, where $c_\beta$ is defined in Theorem 2. Further, assume that $Z_i$s are independent and identically distributed random vectors, with mean 0 and covariance $\Sigma_z \in \mathbb{R}^{p\times p}$. Furthermore, its fourth order moment is finite. The same assumption is also needed for $V_{it}$ across both $1 \leq i \leq N$ and $0 \leq t \leq T$. Moreover, we need $\{Z_i\}$ and $\{U_{it}\}$ to be mutually independent.

(C2) (Network Structure)

(C2.1) (Connectivity) Let the set of all the nodes $\{1, \cdots, N\}$ be the state space of a Markov chain, with the transition probability given by $W$. It is assumed the Markov chain is irreducible and aperiodic. In addition, define $\pi = (\pi_i)^\top \in \mathbb{R}^N$ to be the stationary distribution vector of the Markov chain (i.e., $\pi_i \geq 0$, $\sum_i \pi = 1$, and $W^\top \pi = \pi$). It is assumed that $\sum_{i=1}^N \pi_i^2 \to 0$ as $N \to \infty$.

(C2.2) (Sparsity) Assume $|\lambda_1(W^*)| = \mathcal{O}(\log N)$, where $W^*$ is defined to be a symmetric matrix as $W^* = W + W^\top$.

(C3) (Convergence) Assume $\widehat{\Omega}_1 \xrightarrow{p} \Omega_1$ as $N \to \infty$, where $\Omega_1 = (\Omega_{1,ij}) \in \mathbb{R}^{N\times N}$ is a positive definite matrix. In addition, assume the following limits exist. They are, respectively, $\kappa_1 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(\Sigma_Y)$, $\kappa_2 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(W\Sigma_Y)$, $\kappa_3 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(W\Sigma_Y W^\top)$, and $\kappa_4 = \lim_{N\to\infty} N^{-1}\mathrm{tr}\{(I-G)^{-1}\}$, $\kappa_5 = \lim_{N\to\infty} N^{-1}\mathrm{tr}\{W(I-G)^{-1}\}$. Here $\kappa_j$ $(1 \le j \le 5)$ are fixed constants.

(C4) (Density) There exists positive constants $0 < c_1 < c_2 < \infty$ such that $c_1 \leq f_{it}(x) \leq c_2$ for all $1 \leq i \leq N, 1 \leq t \leq T$ with $x \in \mathbb{R}$.

(C5) (Monotonicity) It is assumed $\theta(\tau)^\top X_{it}$ $(1 \leq i \leq N, 1 \leq t \leq T)$ are monotone increasing functions with respect to $\tau$.

Return to Stationarity Return to An amuse gueule of theory Return to Antipasti Theory

# Expectile as quantile

$e_\tau(Y)$ is the $\tau$-quantile of the cdf $T$, where

$$T(y) = \frac{G(y) - xF(y)}{2\{G(y) - yF(y)\} + \{y - \mu_Y\}}, \tag{10}$$

$$G(y) = \int_{-\infty}^{y} u \, dF(u) \tag{11}$$

▸ Back Quantiles and Expectiles

# Definition of $\Sigma_V$

- $\text{vec}(\Sigma_Y) = (M_1 - c_1^{-2}c_0^2)\mathbf{1}_{N^2} + c_1^{-1}c_0(I - G^*)^{-1}\text{vec}(\Sigma_{bv}) + (I - G^*)^{-1}\text{vec}(\Sigma_V)$
- $c_1 = (1 - b_1 - b_2)^{-1}$,,
- $M_1 = c_1^{-1}c_0^2(1 + b_1 + b_2)(I - G^*)^{-1}$, $\Sigma_{bv} = \sigma_{bv} I_N$,
- $\sigma_{bv} = \mathrm{E}[\{\beta_1(U_{it}) + \beta_2(U_{it})\} V_{it}]$.

Back $\Sigma_V$

# Definition of $\Omega_0$

$$\Omega_0 = \begin{pmatrix} 1 & \mathbf{0}^\top & c_b & c_b \\ \mathbf{0} & \Sigma_z & \kappa_5 \overline{\gamma}^\top \Sigma_z & \kappa_4 \overline{\gamma}^\top \Sigma_z \\ c_b & \kappa_5 \Sigma_z \overline{\gamma}^\top & \kappa_3 + c_b^2 & \kappa_2 + c_b^2 \\ c_b & \kappa_4 \Sigma_z \overline{\gamma}^\top & \kappa_2 + c_b^2 & \kappa_1 + c_b^2 \end{pmatrix} \tag{12}$$

$c_b = c_1^{-1} c_0$, and $\Omega_1$ is defined in condition (C3).

Back $\Omega_0$

# Definitions

- $b_1 = \mathrm{E}\{\beta_1(U_{it})\}$
- $b_2 = \mathrm{E}\{\beta_2(U_{it})\}$
- $c_{\beta_1,\tau_1\tau_2} = \int_{\tau_1}^{\tau_2} \beta_1(u)du$
- $c_{\beta_2,\tau_1\tau_2} = \int_{\tau_1}^{\tau_2} \beta_2(u)du.$

Return to Impulse Analysis

# Gamma distribution

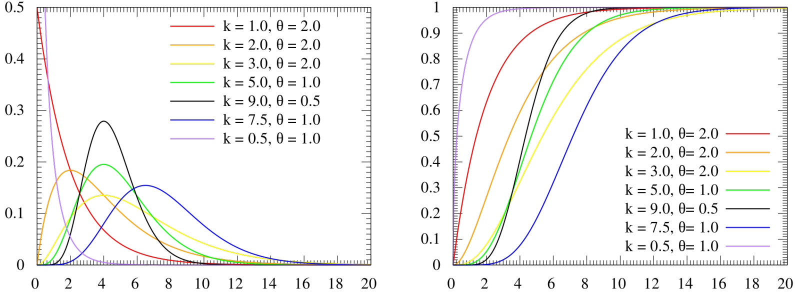


Figure 13: pdf of Gamma distribution (Left) and cdf of Gamma distribution (Right), source: wikipedia.org.

Return to Gamma distribution

# Quantile Regression

$$Y_t = \theta_0(u_t) + \theta_1(u_t) Y_{t-1} + \ldots + \theta_p(u_t) Y_{t-p}$$

$$\begin{aligned} Q_{Y_t}(\tau | Y_{t-1}, \ldots, Y_{t-p}) &= \theta_0(\tau) + \theta_1(\tau) Y_{t-1} + \ldots + \theta_p(\tau) Y_{t-p} \\ &= x_t \top \theta(\tau) \end{aligned}$$

with $x_t = (1, Y_{t-1}, \ldots, Y_{t-p})$

$$Q_{g(u)}(\tau) = g(Q_u(\tau)) = g(\tau)$$

▸ Back to Quantile Regression

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