Introduction to Copulas

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Example

- we pay 200 EUR for the chance to win 1000 EUR, if DAX returns decrease by 2%

$$\mathrm{P}_{DAX}(r_{DAX} \leq -0.02) = F_{DAX}(-0.02) = 0.2$$

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$$\mathrm{P}_{DAX}(r_{DAX} \leq -0.02) = F_{DAX}(-0.02) = 0.2$$

- we pay 200 EUR for the chance to win 1000 EUR, if DJ returns decrease by 1%

$$\mathrm{P}_{DJ}(r_{DJ} \leq -0.01) = F_{DJ}(-0.01) = 0.2$$

Example

- we get 1000 EUR if DAX and DJ indices decrease simultaneously by 2% and 1% respectively.
  how much are we ready to pay in this case?

$$
\begin{aligned}
&\mathrm{P}\{(r_{DAX} \leq -0.02) \wedge (r_{DJ} \leq -0.01)\} \\
&\quad = F_{DAX,DJ}(-0.02, -0.01) \\
&\quad = C\{F_{DAX}(-0.02), F_{DJ}(-0.01)\} \\
&\quad = C(0.2, 0.2).
\end{aligned}
$$

Univariate Case

Let $x_1, \ldots, x_n$ be realizations of the random variable X
$X \sim F$, where F is unknown

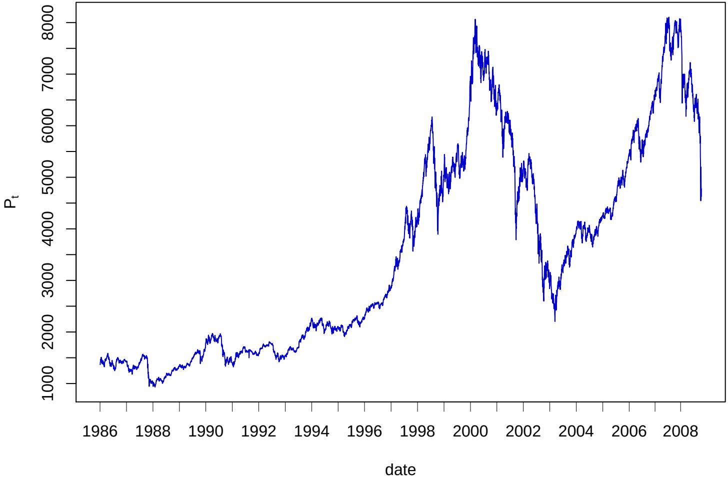
Example 1

- x_i are returns of the asset for one firm at the day t_i
- x_i are numbers of sold albums *The Man Who Sold the World* by *David Bowie* at day t_i

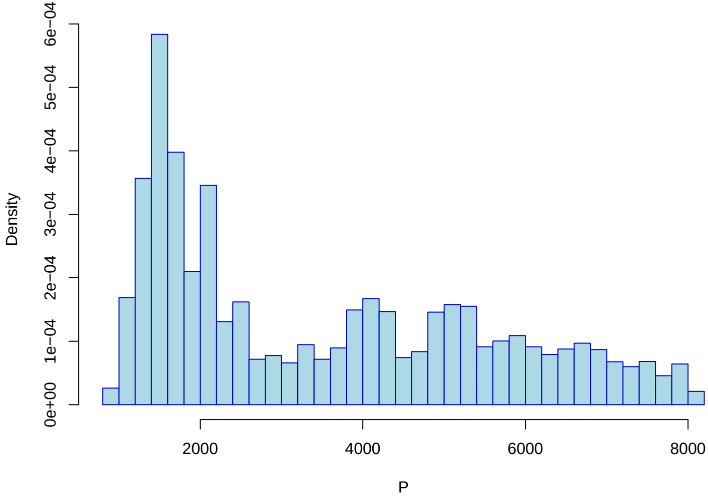
What is a good approximation of F ?

`traditional` or `modern` approach

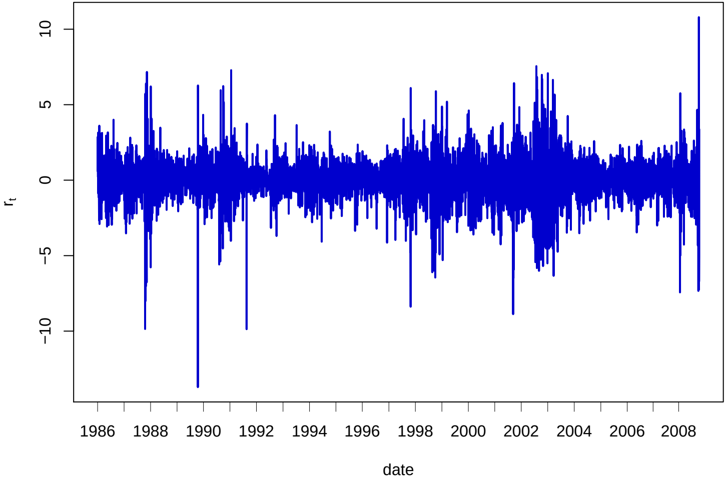
DAX (P_t)

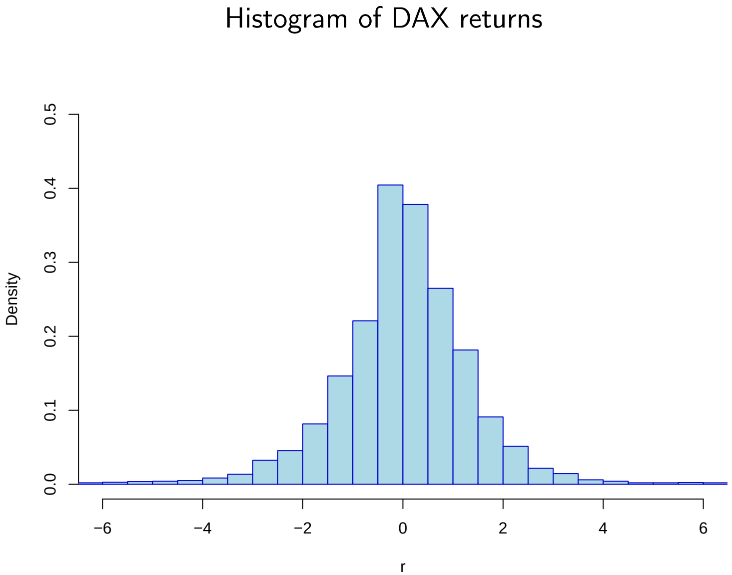

8000
7000
6000
5000
4000
3000
2000
1000
P_t
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
date

Histogram of DAX



DAX returns ($r_t = \log \frac{P_t}{P_{t-1}}$)




Histogram of DAX returns
Density
0.0
0.1
0.2
0.3
0.4
0.5
−6
−4
−2
0
2
4
6
r

Traditional approach:

F_0 – known distribution

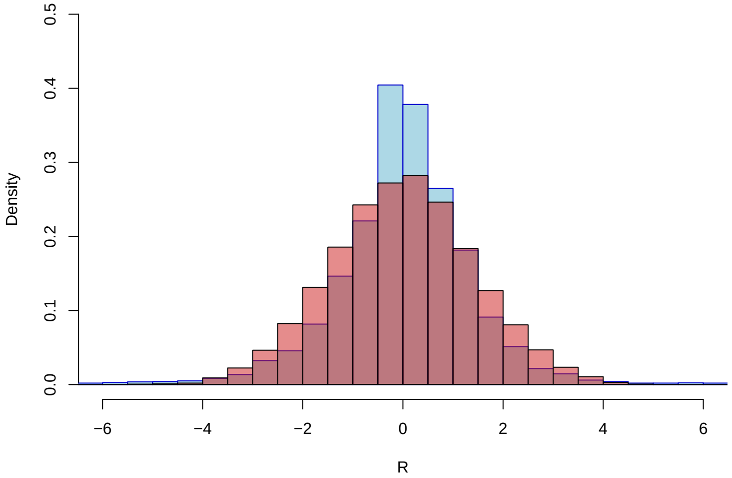
- parameters of F_0 are estimated from the sample $x_1, \ldots, x_n$
  - $F_0 = \mathrm{N}(\mu, \sigma^2) \;\Rightarrow\; (\mu, \sigma)$, here $\hat{\mu} = \bar{x}$, $\hat{\sigma}^2 = \hat{s}^2$
  - $F_0 = St(\alpha, \beta, \mu, \sigma^2) \;\Rightarrow\; (\alpha, \beta, \mu, \sigma)$ are estimated by Hull Estimator, Tail Exponent Estimation, etc.
- check the appropriateness of F_0 by a test (KS type)

$$H_0 : F = F_0 \quad \text{vs} \quad H_1 : F \neq F_0$$

- if test confirm F_0, use $\hat{F}_0$

Fit of the Normal distribution to DAX returns

$(\hat{\mu} = 0.0002113130,\ \hat{\sigma}^2 = 0.0002001865)$



Modern approach: calculate the edf

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}\{X_i \leq x\},$$

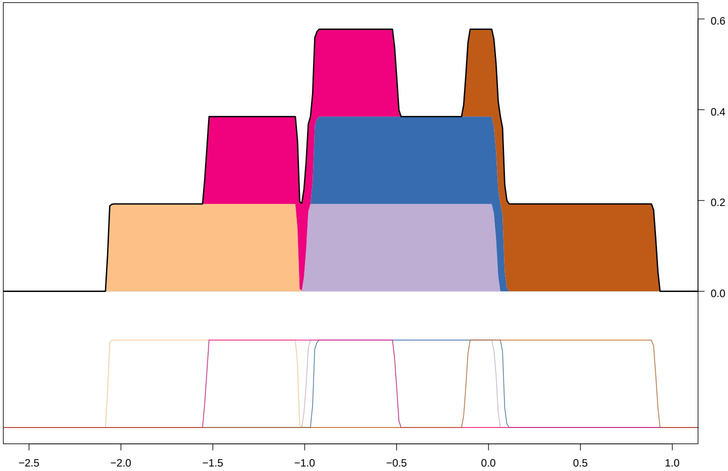
or the nonparametric kernel smoother

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

name	$K(u)$
Uniform	$\frac{1}{2}\mathbf{I}\{\lvert u\rvert \leq 1\}$
Epanechnikov	$\frac{3}{4}(1-u^2)\mathbf{I}\{\lvert u\rvert \leq 1\}$
Gaussian	$\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\}$

Kernel smoothing with UNI kernel

$$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$$



Kernel smoothing with EPA kernel

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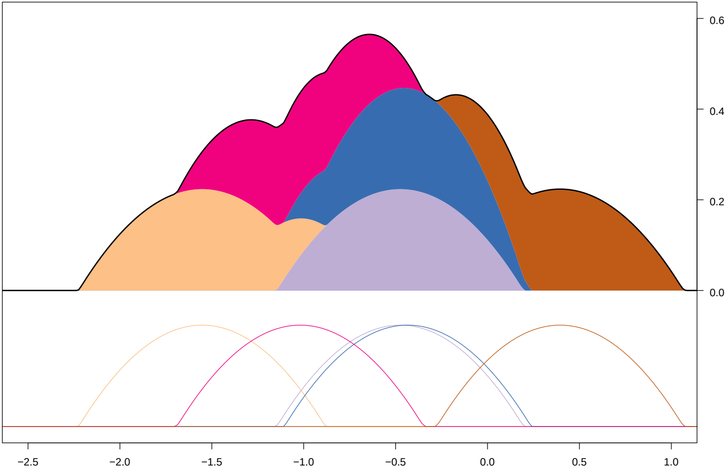
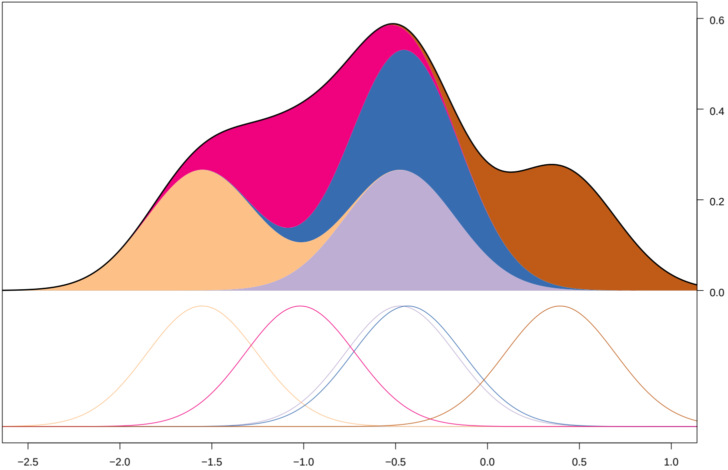
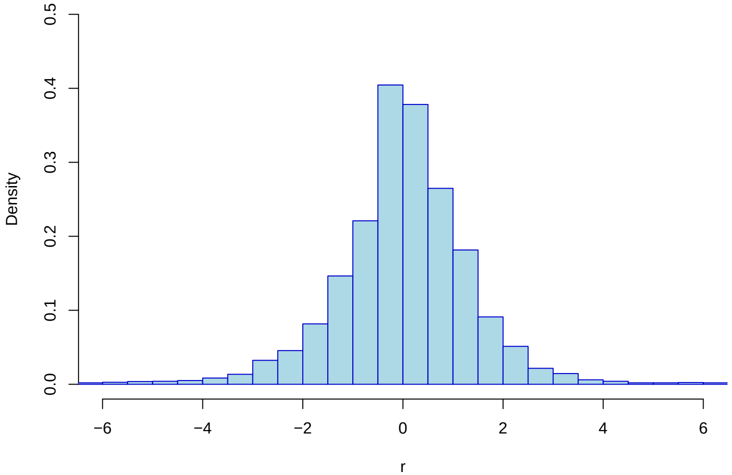


Kernel smoothing with GAU kernel

$$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$$



Histogram of DAX returns



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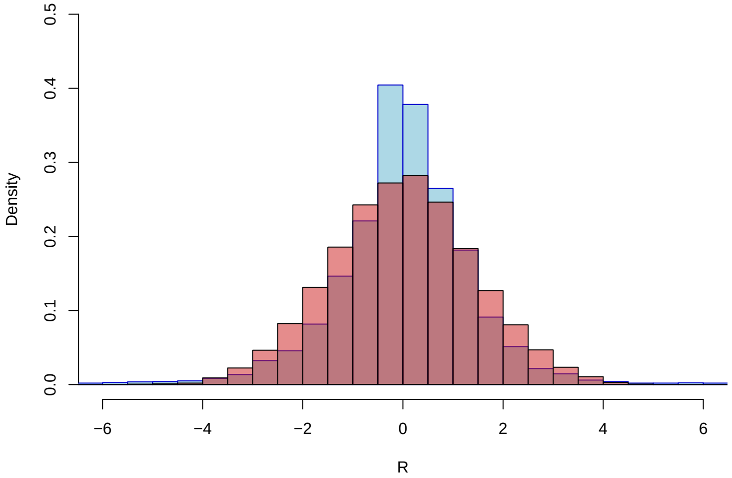

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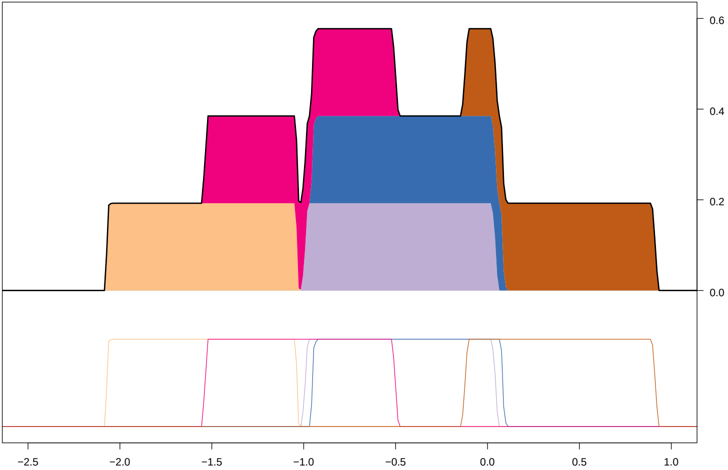

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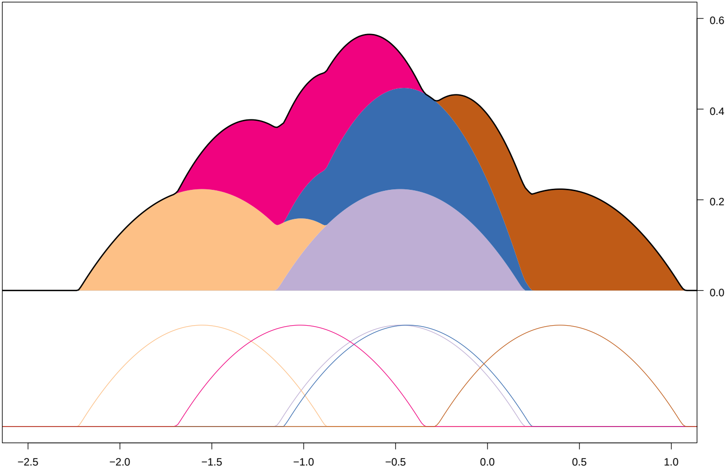

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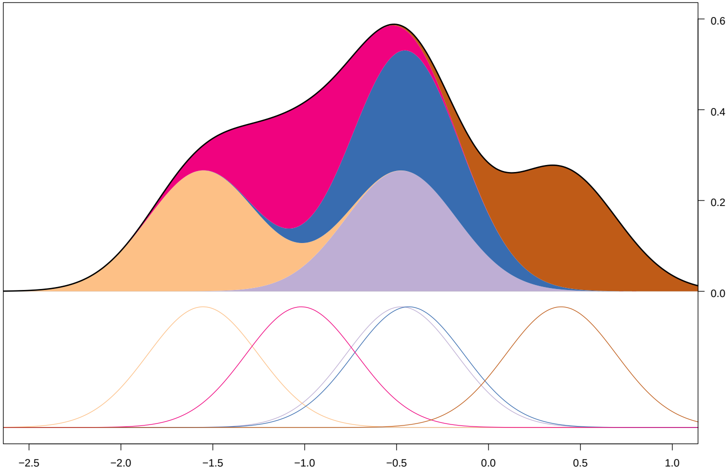
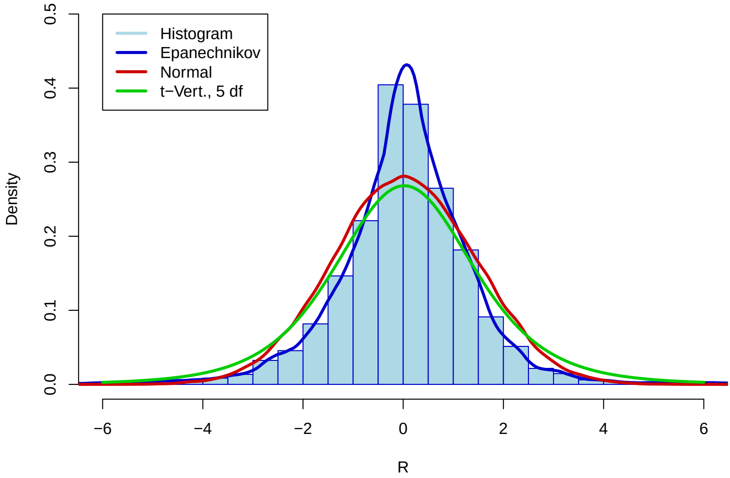


The estimated density of DAX returns



Multivariate Case

$\{x_{1i}, \ldots, x_{di}\}_{i=1,\ldots,n}$ is the realization of the vector $(X_1, \ldots, X_d) \sim \mathbf{F}$, where $\mathbf{F}$ is unknown.

Example 2

- $\{x_{1i}, \ldots, x_{di}\}_{i=1,\ldots,n}$ are returns of the d assets in the portfolio at day t_i
- $(x_{1i}, x_{2i})^\top$ are numbers of sold albums *The Man Who Sold The World* by *David Bowie* and singles *I Saved The World Today* by *Eurythmics* at day t_i

Multivariate Case

What is a good approximation of F ?

`traditional` or `modern` approach

Very flexible approximation to **F** is challenging in high dimension due to curse of dimensionality.

Traditional approach: Mainly restricted to the class of elliptical distributions: Normal or t distributions

$$f_N(x_1,\dots,x_d) = \frac{1}{\sqrt{|\Sigma|(2\pi)^d}} \exp\left\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}$$

Drawbacks of the elliptical distributions:

1. does not often describe financial data properly
2. huge number of parameters to be estimated

   f.e. for Normal distribution: $\underbrace{\frac{d(d-1)}{2}}_{in\ dependency} + \underbrace{2d}_{in\ margins}$

3. ellipticity

Simulate $X \sim \mathrm{N}(\mu, \Sigma)$ with the sample size $n = 1000$ and estimate the parameters $(\hat{\mu}, \hat{\Sigma})$

$$\Sigma = \begin{pmatrix} 1.5 & 0.7 & 0.2 \\ 0.7 & 1.3 & -0.4 \\ 0.2 & -0.4 & 0.3 \end{pmatrix} \Rightarrow \hat{\Sigma} = \begin{pmatrix} 1.461 & 0.726 & 0.181 \\ 0.726 & 1.335 & -0.408 \\ 0.181 & -0.408 & 0.301 \end{pmatrix}$$

$$\mu = (0,\ 0,\ 0) \Rightarrow \hat{\mu} = (0.0175,\ -0.0022,\ 0.0055)$$

$\hat{\Sigma}$ and Σ are not close to each other for only 3 dimensions and quiet big sample

Correlation

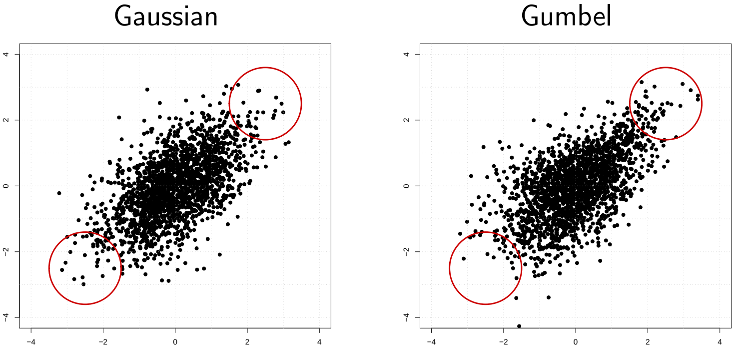


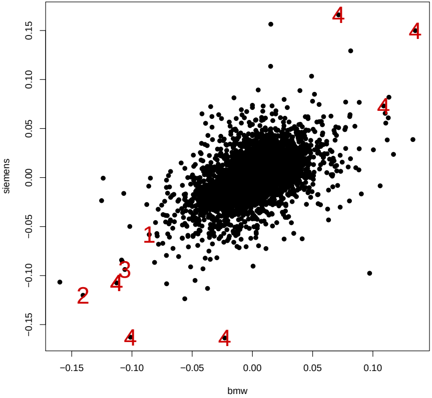
Abbildung 1: Scatterplots for two distribution with $\rho = 0.4$

- same marginal distributions
- same linear correlation coefficient

"Extreme, **synchronized rises and falls** in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - **the "perfect storm" scenario**"

(Business Week, September 1998)

Correlation

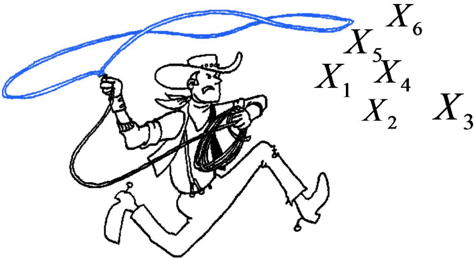


1. 19.10.1987
   Black Monday
2. 16.10.1989
   Berlin Wall
3. 19.08.1991
   Kremlin
4. 17.03.2008, 19.09.2008, 10.10.2008, 13.10.2008, 15.10.2008, 29.10.2008
   Crisis

Copula

For a distribution function F with marginals $F_{X_1}, \dots, F_{X_d}$, there exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \dots, x_d) = \mathbf{C}\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\}.$$



A little bit of history

- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions

1914–91, b. Mustamäki, Finland; d. Chapel Hill, NC
gained his PhD from U Berlin in 1940
1924–45 work in U Berlin

Wassilij Hoeffding on BBI

A little bit of history

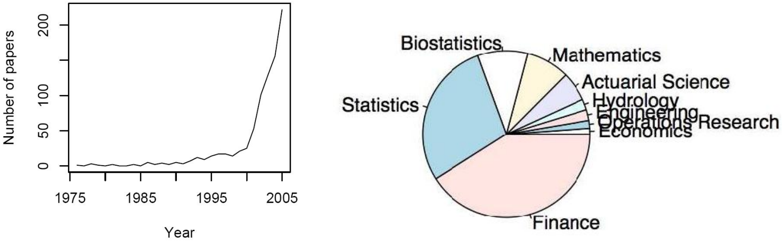
- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions
- 1959: The word **copula** appears for the first time (*Abe Sklar*)
- 1999: Introduced to financial applications (*Paul Embrechts*, *Alexander McNeil*, *Daniel Straumann* in RISK Magazine)
- 2000: Paper by *David Li* in *Journal of Derivatives* on application of copulae to CDO
- 2006: Several insurance companies, banks and other financial institutions apply copulae as a risk management tool

Applications

Practical Use:

1. medicine (Vandenhende (2003))
2. hydrology (Genest and Favre (2006))
3. biometrics (Wang and Wells (2000, JASA), Chen and Fan (2006, CanJoS))
4. economics
   - portfolio selection (Patton (2004, JoFE), Xu (2004, PhD thesis), Hennessy and Lapan (2002, MathFin))
   - time series (Chen and Fan (2006a, 2006b, JoE), Fermanian and Scaillet (2003, JoR), Lee and Long (2005, JoE))
   - risk management (Junker and May (2002, EJ), Breyman et. al. (2003, QF))

Applications



Bourdeau-Brien (2007) covers 871 publications

Special Copulas

Theorem

Let C be a copula. Then for every $(u_1, u_2) \in [0,1]^2$

$$\max(u_1 + u_2 - 1, 0) \leq C(u_1, u_2) \leq \min(u_1, u_2),$$

where bounds are called ***lower and upper Fréchet-Höffdings bounds****. When they are copulas they represent perfect negative and positive dependence respectively.*

The simplest copula is **product copula**

$$\Pi(u_1, u_2) = u_1 u_2$$

characterize the case of independence.

Copula Classes

1. elliptical
    - implied by well-known multivariate df's (Normal, t), derived through Sklar's theorem
    - do not have closed form expressions and are restricted to have radial symmetry
2. Archimedean

$$C(u_1, u_2) = \phi^{-1}\{\phi(u_1) + \phi(u_2)\}$$

- allow for a great variety of dependence structures
- closed form expressions
- several useful methods for multivariate extension
- not derived from mv df's using Sklar's theorem

Copula Examples 1

Gaussian copula

$$
\begin{aligned}
C_\delta^G(u_1, u_2) &= \boldsymbol{\Phi}_\delta\{\boldsymbol{\Phi}^{-1}(u_1), \boldsymbol{\Phi}^{-1}(u_2)\} \\
&= \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_1)} \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\delta^2}} \exp\left\{\frac{-(s^2 - 2\delta st + t^2)}{2(1-\delta^2)}\right\} ds\, dt,
\end{aligned}
$$

- Gaussian copula contains the dependence structure
- *normal* marginal distribution + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distribution + Gaussian copula = meta-Gaussian distributions
- allows to generate joint symmetric dependence, but no tail dependence

Copula Examples 2

Gumbel copula

$$C_\theta^{Gu}(u_1, u_2) = \exp\left\{-\left[(-\log u_1)^{1/\theta} + (-\log u_2)^{1/\theta}\right]^\theta\right\}.$$

- for $\theta > 1$ allows to generate dependence in the upper tail
- for $\theta = 1$ reduces to the product copula
- for $\theta \to \infty$ obtain Frèchet-Hoeffding upper bound

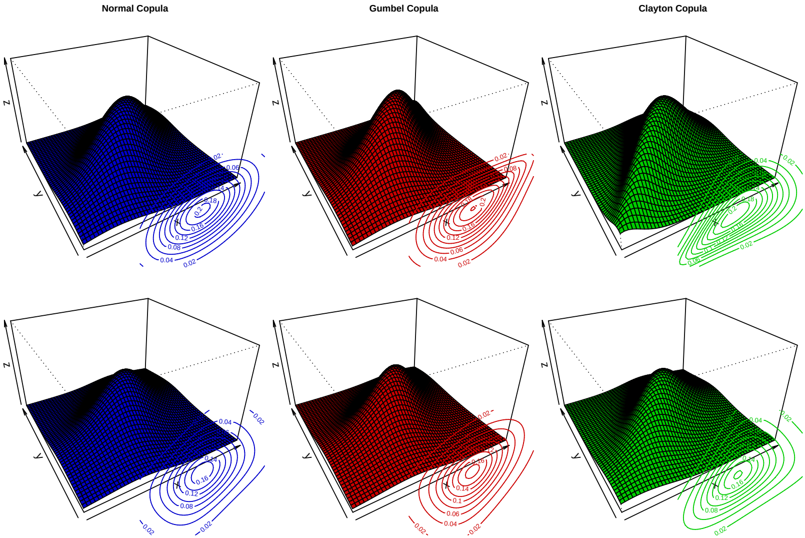
$$C_\theta(u_1, u_2) \stackrel{\theta\to\infty}{\to} \min(u_1, u_2)$$

Copula Examples 3

Clayton copula

$$C_\theta^{Cl}(u_1, u_2) = \left[\max\left(u_1^{-\theta} + u_2^{-\theta} - 1, 0\right)\right]^{-\frac{1}{\theta}}$$

- dependence becomes maximal when $\theta \to \infty$
- independence is achieved when $\theta = 0$
- the distribution tends to the lower Frèchet-Hoeffding bound when $\theta \to 1$
- allows to generate asymmetric dependence and lower tail dependence, but no upper tail dependence


Normal Copula
Gumbel Copula
Clayton Copula

Dependencies, Linear Correlation

$$\delta(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}}.$$

- Sensitive to outliers
- Measures the 'average dependence' between X_1 and X_2
- Invariant under strictly increasing linear transformations
- May be misleading in situations where multivariate df is not elliptical

Dependencies, Kendall's tau

Definition

If F is continuous bivariate cdf and let (X_1, X_2), (X_1', X_2') be independent random pairs with distribution F. Then **Kendall's tau** is

$$\tau = P\{(X_1 - X_1')(X_2 - X_2') > 0\} - P\{(X_1 - X_1')(X_2 - X_2') < 0\}$$

- Less sensitive to outliers
- Measures the 'average dependence' between X and Y
- Invariant under strictly increasing transformations
- Depends only on the copula of (X_1, X_2)
- For elliptical copulae: $\delta(X_1, X_2) = \sin\left(\frac{\pi}{2}\tau\right)$

Dependencies, Spearmans's rho

Definition

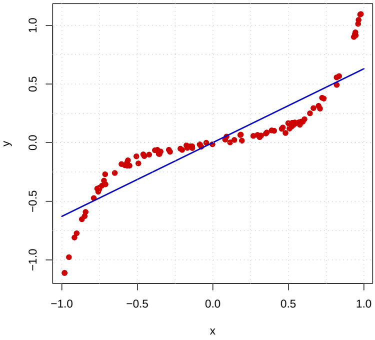
If F is a continuous bivariate cumulative distribution function with marginal F_1 and F_2 and let $(X_1, X_2) \sim F$. Then **Spearmans's rho** is a correlation between $F_1(X_1)$ and $F_2(X_2)$

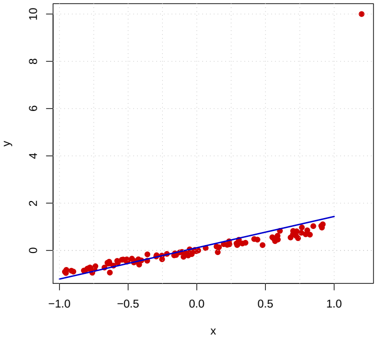
$$\rho = \frac{Cov\{F_1(X_1), F_2(X_2)\}}{\sqrt{Var\{F_1(X_1)\}Var\{F_2(X_2)\}}}.$$

- Less sensitive to outliers
- Measures the 'average dependence' between X_1 and X_2
- Invariant under strictly increasing transformations
- Depends only on the copula of (X_1, X_2)
- For elliptical copulae: $\delta(X_1, X_2) = 2\sin\left(\frac{\pi}{6}\rho\right)$

$$\delta = 0.892, \quad \tau = 0.956, \quad \rho = 0.996$$

$$\delta = 0.659, \quad \tau = 0.888, \quad \rho = 0.982$$





Dependencies, Examples

Gaussian copula

$$\begin{aligned} \rho &= \frac{6}{\pi} \arcsin \frac{\delta}{2}, \\ \tau &= \frac{2}{\pi} \arcsin \delta, \end{aligned}$$

where δ is a linear correlation coefficient.
Gumbel copula

$$\begin{aligned} \rho &- \text{no closed form}, \\ \tau &= 1 - \frac{1}{\theta}. \end{aligned}$$

Multivariate Copula Definition

Definition

The **copula** is a multivariate distribution with all univariate margins being $U(0,1)$.

Theorem (Sklar, 1959)

Let $X_1, \ldots, X_k$ *be random variables with marginal distribution functions* $F_1, \ldots, F_k$ *and joint distribution function* F. *Then there exists a* k*-dimensional copula* $C : [0,1]^k \to [0,1]$ *such that* $\forall x_1, \ldots, x_k \in \overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \ldots, x_k) = C\{F_1(x_1), \ldots, F_k(x_k)\} \qquad (1)$$

If the margins $F_1, \ldots, F_k$ *are continuous, then* C *is unique. Otherwise* C *is uniquely determined on* $F_1(\overline{\mathbb{R}}) \times \cdots \times F_k(\overline{\mathbb{R}})$ *Conversely, if* C *is a copula and* $F_1, \ldots, F_k$ *are distribution functions, then the function* F *defined in (1) is a joint distribution function with margins* $F_1, \ldots, F_k$.

Copula Density

Several theorems provides existence of derivatives of copulas, having them copula density is defined as

$$c(u_1, \dots, u_k) = \frac{\partial^n C(u_1, \dots, u_k)}{\partial u_1 \dots \partial u_k}.$$

Joint density function based on copula

$$_c f(x_1, \dots, x_k) = c\{F_1(x_1), \dots, F_k(x_k)\} \cdot f_1(x_1) \dots f_k(x_k),$$

where $f_1(\cdot), \dots, f_k(\cdot)$ are marginal density functions.

Special Copulas

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$$\max\left(\sum_{i=1}^{k} u_i + 1 - k, 0\right) \leq C(u_1, \ldots, u_k) \leq \min(u_1, \ldots, u_k),$$

where bounds are called ***lower and upper Fréchet-Höffdings bounds****. When they are copulas they represent perfect negative and positive dependence respectively.*

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$$\Pi(u_1, \ldots, u_k) = \prod_{i=1}^{k} u_i$$

characterize the case of independence.

Simulation

Frees and Valdez, (1998, NAAJ), Whelan, (2004, QF), Marshal and Olkin, (1988, JASA)

Conditional inversion method:

Let $C = C(u_1, \dots, u_k)$, $C_i = C(u_1, \dots, u_i, 1, \dots, 1)$ and $C_k = C(u_1, \dots, u_k)$. Conditional distribution of U_i is given by

$$\begin{aligned} C_i(u_i|u_1, \dots, u_{i-1}) &= P\{U_i \leq u_i | U_1 = u_1 \dots U_{i-1} = u_{i-1}\} \\ &= \frac{\partial^{i-1} C_i(u_1, \dots, u_i)}{\partial u_1 \dots \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C_{i-1}(u_1, \dots, u_{i-1})}{\partial u_1 \dots \partial u_{i-1}} \end{aligned}$$

- Generate i.r.v. $v_1, \dots, v_k \sim U(0,1)$
- Set $u_1 = v_1$
- $u_i = C_k^{-1}(v_i | u_1, \dots, u_{i-1}) \; \forall i = \overline{2, k}$

Estimations: Empirical Copula

Let $(x^i_{(1)}, \dots, x^i_{(T)})$ be the order statistics if i-th stock and $(r^i_1, \dots, r^i_T)$ corresponding rank statistics such that $x_{(r^i_t)^i} = x^i_t$ for all $i = 1, \dots, d$. Any function

$$\hat{C}\left(\frac{t_1}{T}, \dots, \frac{t_d}{T}\right) = \frac{1}{T}\sum_{t=1}^{T}\prod_{i=1}^{d} \mathbf{I}\{r^i_t \leq t_i\}$$

is an empirical copula

Estimation: bivariate case

- based on Kendall's τ estimator

$$\tau_n = \frac{4}{n(n-1)} P_n - 1,$$

where P_n is the number of concordant pairs.
For Gumbel copula $\widehat{\theta}_n = \frac{1}{1-\tau_n}$

- based on Spearman's ρ estimator

$$\rho_n = \frac{\sum_{i=1}^{n} (R_i - \overline{R})^2 (S_i - \overline{S})^2}{\sqrt{\sum_{i=1}^{n} (R_i - \overline{R})^2 \sum_{i=1}^{n} (S_i - \overline{S})^2}},$$

where $(R_i, S_i)\ \forall i = \overline{1, n}$ are pairs of ranks.
For Gaussian Copula $\delta_n = 2 \sin \frac{\pi \rho_n}{6}$

Copula Estimation

The distribution of $X = (X_1, \ldots, X_d)'$ with marginals $F_{X_j}(x_j, \delta_j)$ $j = 1, \ldots, d$ is given by

$$F_X(x_1, \ldots, x_d) = C\{F_{X_1}(x_1, \delta_1), \ldots, F_{X_d}(x_d, \delta_d); \theta\}$$

and its density is given by

$$f(x_1, \ldots, x_d, \delta_1, \ldots, \delta_d, \theta) = c\{F_{X_1}(x_1, \delta_1), \ldots, F_{X_d}(x_d, \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j, \delta_j)$$

Copula Estimation

For a sample of observations $\{x_t\}_{t=1}^{\prime}$ and $\vartheta = (\delta_1, \ldots, \delta_d; \theta) \in \mathbb{R}^{d+1}$ the likelihood function is

$$L(\vartheta; x_1, \ldots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \ldots, x_{d,t}; \delta_1, \ldots, \delta_d; \theta)$$

and the corresponding log-likelihood function

$$\begin{aligned} \ell(\vartheta; x_1, \ldots, x_T) &= \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}, \delta_1), \ldots, F_{X_d}(x_{d,t}, \delta_d); \theta\} \\ &+ \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{j,t}, \delta_j) \end{aligned}$$

Full Maximum Likelihood (FML)

- FML estimates vector of parameters ϑ in one step through

$$\tilde{\vartheta}_{FML} = \arg\max_{\vartheta} \ell(\vartheta)$$

- the estimates $\tilde{\vartheta}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})'$ solve

$$(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta)' = 0$$

- Drawback:with an increasing dimension the algorithm becomes too burdensome computationally

Inference for Margins (IFM)

1. estimate parameters δ_j from the marginal distributions:

$$\hat{\delta}_j = \arg\max_{\delta} \left\{ \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j) \right\}$$

2. estimate the dependence parameter θ by minimizing the *pseudo log-likelihood* function

$$\ell(\theta; \hat{\delta}_1, \dots, \hat{\delta}_d) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \dots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}$$

3. the estimates $\hat{\vartheta}_{IFM} = (\hat{\delta}_1, \dots, \hat{\delta}_d, \hat{\theta})'$ solve

$$(\partial\ell/\partial\delta_1, \dots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta)' = 0$$

4. Advantage: numerically stable

Canonical Maximum Likelihood (CML)

- CML maximizes the *pseuso log-likelihood* function with *empirical* marginal distributions

$$\begin{aligned} \ell(\theta) &= \sum_{t=1}^{T} \log c\{\hat{F}_{X_1}(x_{1,t}), \ldots, \hat{F}_{X_d}(x_{d,t}); \theta\} \\ \hat{\vartheta}_{CML} &= \arg\max_{\theta} \ell(\theta) \end{aligned}$$

where

$$\hat{F}_{X_j}(x) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbf{I}\{X_j, t \leq x\}$$

- Advantage: no assumptions about the parametric form of the marginal distributions

$(X_1, X_2) \sim C_\theta^{Gu}$, with $\theta = 1.5$ and
$F_1 = F_2 = \mathcal{N}(\mu_1, \sigma_1^2) = \mathcal{N}(\mu_2, \sigma_2^2) = \mathcal{N}(0, 1)$

	estimate	std. error
μ_1	0.00365	0.00998
σ_1^2	1.00553	0.00690
μ_2	-0.00106	0.00991
σ_2^2	0.99779	0.00684
θ	1.49632	0.01327

Attractive Features

- A copula describes how the marginals are tied together in the joint distribution
- The joint df is decomposed into the marginal dfs and a copula
- The marginal dfs and the copula can be modelled and estimated separately, independent of each other
- Given a copula, we can obtain many multivariate distributions by selecting different marginal dfs
- The copula is invariant under increasing and continuous transformations