Qua de causa copula me placent: Statistics of joint events

Wolfgang K. Härdle
Ostap Okhrin

Institute for Statistics and Econometrics
Humboldt-Universität zu Berlin
http://ise.wiwi.hu-berlin.de

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09

In the mid-'80s, *Wall Street* turned to the quants – *brainy financial engineers* – to invent new ways to boost profits.

Their methods for minting money worked brilliantly...

until one of the them devastated the global economy.

Here's what killed your 401(k). *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick and fatally flawed way to assess risk.*

$$\Pr[T_A < 1, T_B < 1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$$

Probability - Specifically, this is a joint default probability–the likelihood that any two members of the pool (A and B) will both default. It's what investors are looking for, and the rest of the formula provides the answer.

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Survival times - The amount of time between now and when A and B can be expected to default. Li took the idea from a concept in actuarial science that charts what happens to someone's life expectancy when their spouse dies.

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$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Distribution functions - The probabilities of how long A and B are likely to survive. Since these are not certainties, they can be dangerous: Small miscalculations may leave you facing much more risk than the formula indicates.

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$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Copula - This couples (hence the Latinate term copula) the individual probabilities associated with A and B to come up with a single number. Errors here massively increase the risk of the whole equation blowing up.

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$$\Pr[T_A < 1, T_B < 1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Gamma - The all-powerful correlation parameter, which reduces correlation to a single constant-something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.

Example

- we pay 200 EUR for the chance to win 1000 EUR, if DAX returns decrease by 2%

$$\mathrm{P}_{DAX}(r_{DAX} \leq -0.02) = F_{DAX}(-0.02) = 0.2$$

Example

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$$\mathrm{P}_{DAX}(r_{DAX} \leq -0.02) = F_{DAX}(-0.02) = 0.2$$

- we pay 200 EUR for the chance to win 1000 EUR, if DJ returns decrease by 1%

$$\mathrm{P}_{DJ}(r_{DJ} \leq -0.01) = F_{DJ}(-0.01) = 0.2$$

Example

- we get 1000 EUR if DAX and DJ indices decrease simultaneously by 2% and 1% respectively.
  how much are we ready to pay in this case?

$$
\begin{aligned}
\mathrm{P}\{(r_{DAX} \leq -0.02) \wedge (r_{DJ} \leq -0.01)\} \\
&= F_{DAX,DJ}(-0.02, -0.01) \\
&= C\{F_{DAX}(-0.02), F_{DJ}(-0.01)\} \\
&= C(0.2, 0.2).
\end{aligned}
$$

Motivation

- How these colors influence the Pricing?
- How to model Dependency?
- Are there any improvements of Li's model?

Outline

Univariate Case

Let $x_1, \ldots, x_n$ be realizations of the random variable X
$X \sim F$, where F is unknown

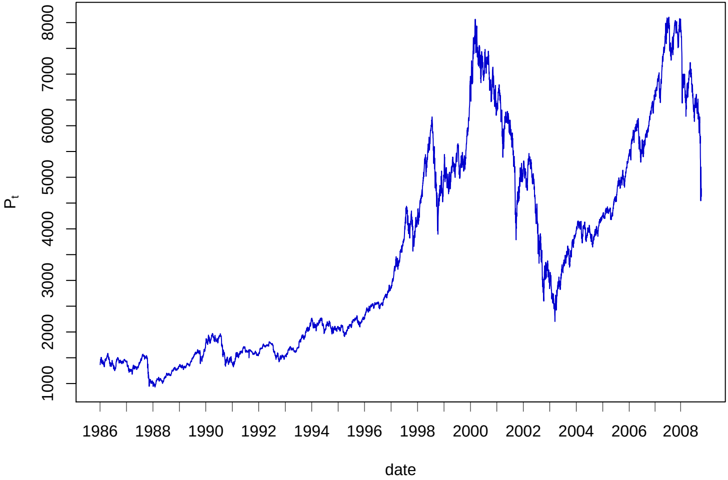
Example 1

- x_i are returns of the asset for one firm at the day t_i
- x_i are numbers of sold albums *The Man Who Sold the World* by *David Bowie* at day t_i

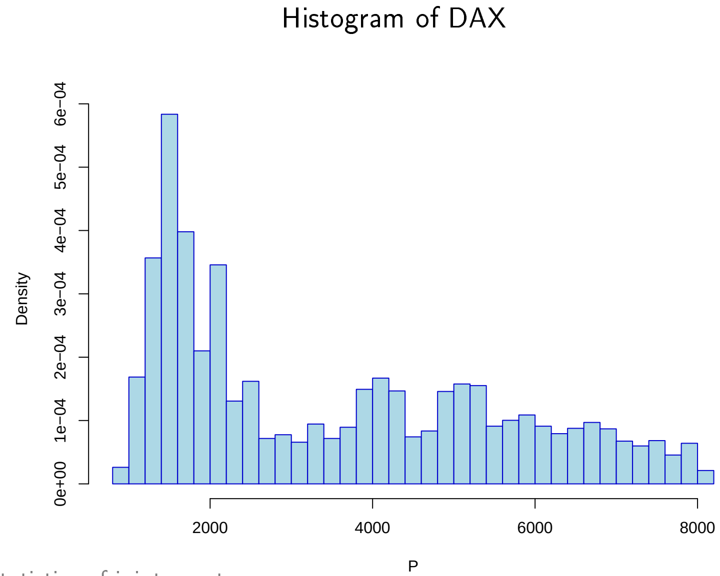
What is a good approximation of F ?

`traditional` or `modern` approach

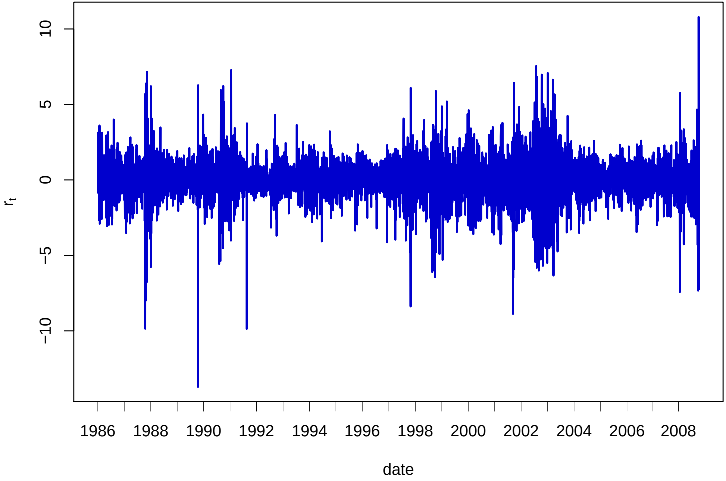
DAX (P_t)



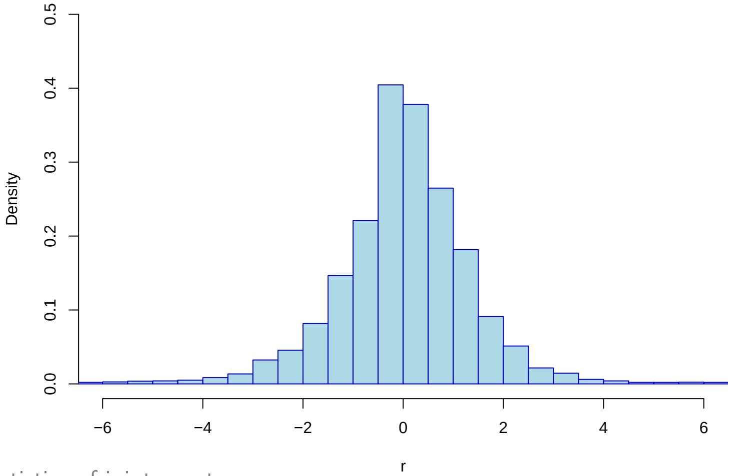
P_t
1000
2000
3000
4000
5000
6000
7000
8000
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
date


Histogram of DAX
Density
0e+00
1e−04
2e−04
3e−04
4e−04
5e−04
6e−04
2000
4000
6000
8000
P

DAX returns ($r_t = \log \frac{P_t}{P_{t-1}}$)



Histogram of DAX returns



Traditional approach:

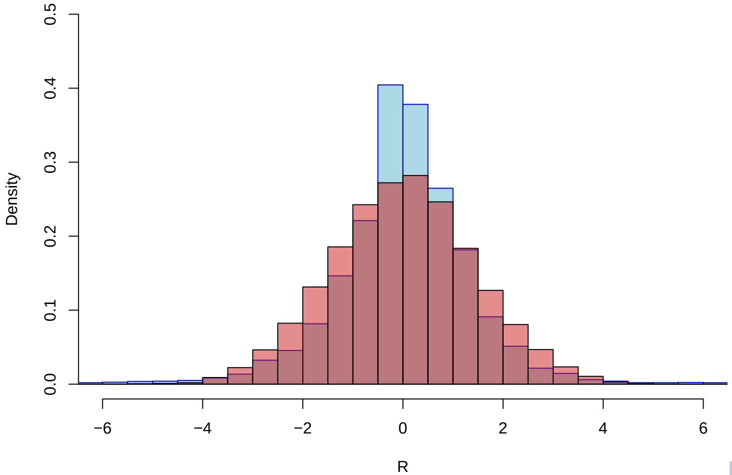
F_0 – known distribution

- parameters of F_0 are estimated from the sample $x_1, \ldots, x_n$
  - $F_0 = \mathrm{N}(\mu, \sigma^2) \;\Rightarrow\; (\mu, \sigma)$, here $\widehat{\mu} = \overline{x}$, $\widehat{\sigma}^2 = \widehat{s}^2$
  - $F_0 = St(\alpha, \beta, \mu, \sigma^2) \;\Rightarrow\; (\alpha, \beta, \mu, \sigma)$ are estimated by Hull Estimator, Tail Exponent Estimation, etc.
- check the appropriateness of F_0 by a test (KS type)

$$H_0 : F = F_0 \quad \text{vs} \quad H_1 : F \neq F_0$$

- if test confirm F_0, use $\widehat{F_0}$

Fit of the Normal distribution to DAX returns
$(\widehat{\mu} = 0.0002113130,\ \widehat{\sigma}^2 = 0.0002001865)$



Modern approach: calculate the edf

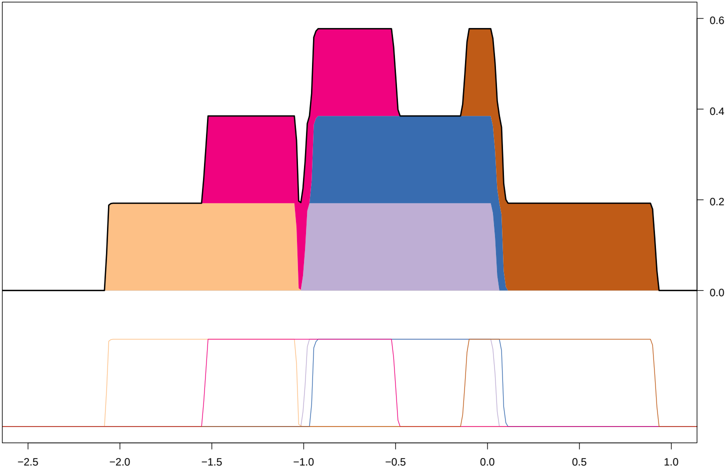
$$\widehat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}\{X_i \leq x\},$$

or the nonparametric kernel smoother

$$\widehat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

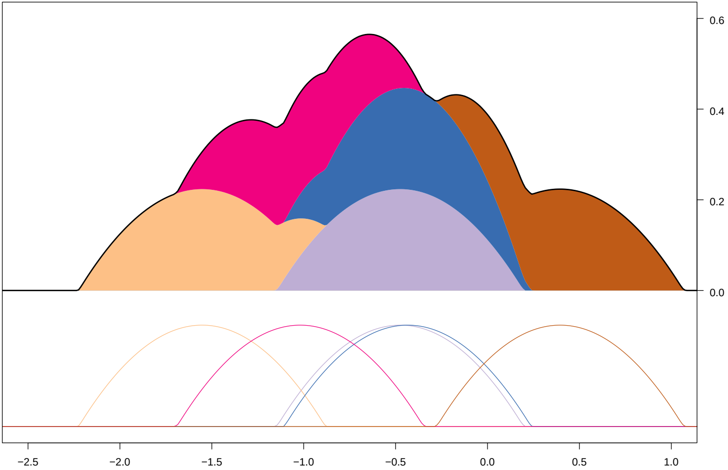
name	$K(u)$
Uniform	$\frac{1}{2}\mathbf{I}\{\lvert u \rvert \leq 1\}$
Epanechnikov	$\frac{3}{4}(1-u^2)\mathbf{I}\{\lvert u \rvert \leq 1\}$
Gaussian	$\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\}$

Kernel smoothing with UNI kernel
$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$



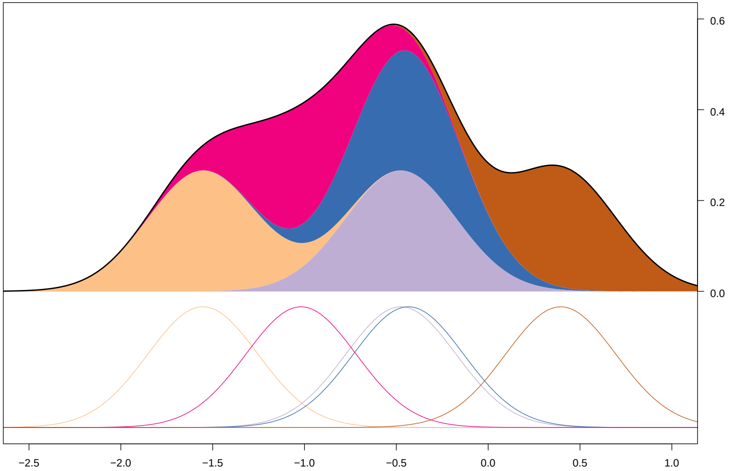
Kernel smoothing with EPA kernel

$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$

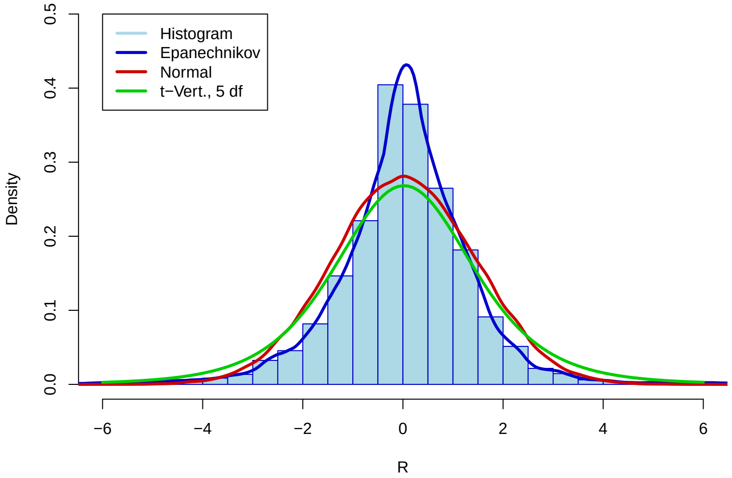


Kernel smoothing with GAU kernel

$$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$$



The estimated density of DAX returns



Multivariate Case

$\{x_{1i}, \dots, x_{di}\}_{i=1,\dots,n}$ is the realization of the vector $(X_1, \dots, X_d) \sim \mathbf{F}$, where $\mathbf{F}$ is unknown.

Example 2

- $\{x_{1i}, \dots, x_{di}\}_{i=1,\dots,n}$ are returns of the d assets in the portfolio at day t_i
- $(x_{1i}, x_{2i})^\top$ are numbers of sold albums *The Man Who Sold The World* by *David Bowie* and singles *I Saved The World Today* by *Eurythmics* at day t_i

Multivariate Case

What is a good approximation of F ?

`traditional` or `modern` approach

Very flexible approximation to **F** is challenging in high dimension due to curse of dimensionality.

Traditional approach: Mainly restricted to the class of elliptical distributions: Normal or t distributions

$$f_N(x_1,\dots,x_d) = \frac{1}{\sqrt{|\Sigma|(2\pi)^d}} \exp\left\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}$$

Drawbacks of the elliptical distributions:

1. does not often describe financial data properly
2. huge number of parameters to be estimated

   f.e. for Normal distribution: $\underbrace{\frac{d(d-1)}{2}}_{in\ dependency} + \underbrace{2d}_{in\ margins}$

3. ellipticity

Simulate $X \sim \mathrm{N}(\mu, \Sigma)$ with the sample size $n = 1000$ and estimate the parameters $(\widehat{\mu}, \widehat{\Sigma})$

$$\Sigma = \begin{pmatrix} 1.5 & 0.7 & 0.2 \\ 0.7 & 1.3 & -0.4 \\ 0.2 & -0.4 & 0.3 \end{pmatrix} \Rightarrow \widehat{\Sigma} = \begin{pmatrix} 1.461 & 0.726 & 0.181 \\ 0.726 & 1.335 & -0.408 \\ 0.181 & -0.408 & 0.301 \end{pmatrix}$$

$$\mu = (0,\ 0,\ 0) \Rightarrow \widehat{\mu} = (0.0175,\ -0.0022,\ 0.0055)$$

$\widehat{\Sigma}$ and Σ are not close to each other for only 3 dimensions and quiet big sample

Correlation

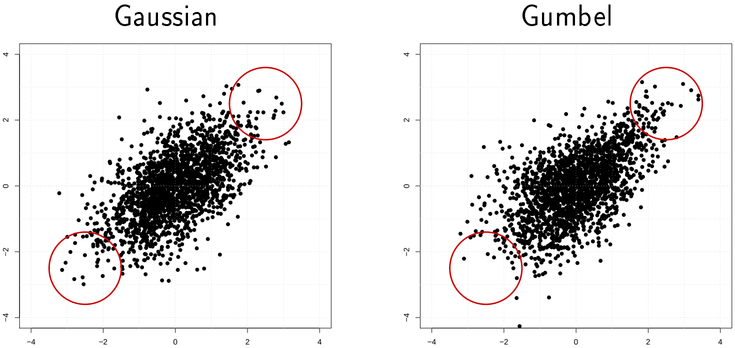


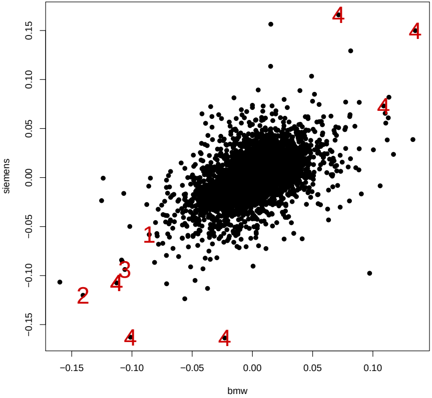
Figure 1: Scatterplots for two distribution with $\rho = 0.4$

- same marginal distributions
- same linear correlation coefficient

"Extreme, **synchronized rises and falls** in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - **the "perfect storm" scenario**"

(Business Week, September 1998)

Correlation

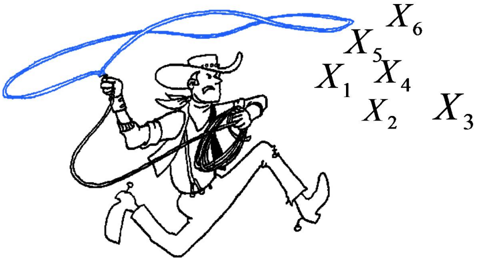


1. 19.10.1987
   Black Monday
2. 16.10.1989
   Berlin Wall
3. 19.08.1991
   Kremlin
4. 17.03.2008, 19.09.2008, 10.10.2008, 13.10.2008, 15.10.2008, 29.10.2008
   Krise

Copula

For a distribution function F with marginals $F_{X_1}, \ldots, F_{X_d}$, there exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \ldots, x_d) = \mathbf{C}\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}.$$



A little bit of history

- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions

1914–91, b. Mustamäki, Finland; d. Chapel Hill, NC
gained his PhD from U Berlin in 1940
1924–45 work in U Berlin

Wassilij Hoeffding on BBI

A little bit of history

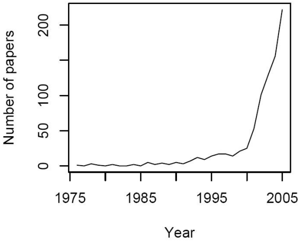
- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions
- 1959: The word **copula** appears for the first time (*Abe Sklar*)
- 1999: Introduced to financial applications (*Paul Embrechts, Alexander McNeil, Daniel Straumann* in RISK Magazine)
- 2000: Paper by *David Li* in *Journal of Derivatives* on application of copulae to CDO
- 2006: Several insurance companies, banks and other financial institutions apply copulae as a risk management tool

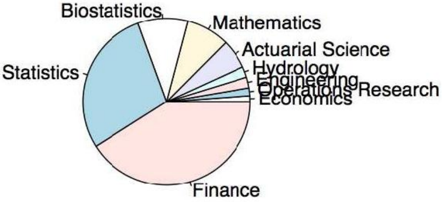
Applications

Practical Use:

1. medicine (Vandenhende (2003))
2. hydrology (Genest and Favre (2006))
3. biometrics (Wang and Wells (2000, JASA), Chen and Fan (2006, CanJoS))
4. economics
    - portfolio selection (Patton (2004, JoFE), Xu (2004, PhD thesis), Hennessy and Lapan (2002, MathFin))
    - time series (Chen and Fan (2006a, 2006b, JoE), Fermanian and Scaillet (2003, JoR), Lee and Long (2005, JoE))
    - risk management (Junker and May (2002, EJ), Breyman et. al. (2003, QF))

Applications





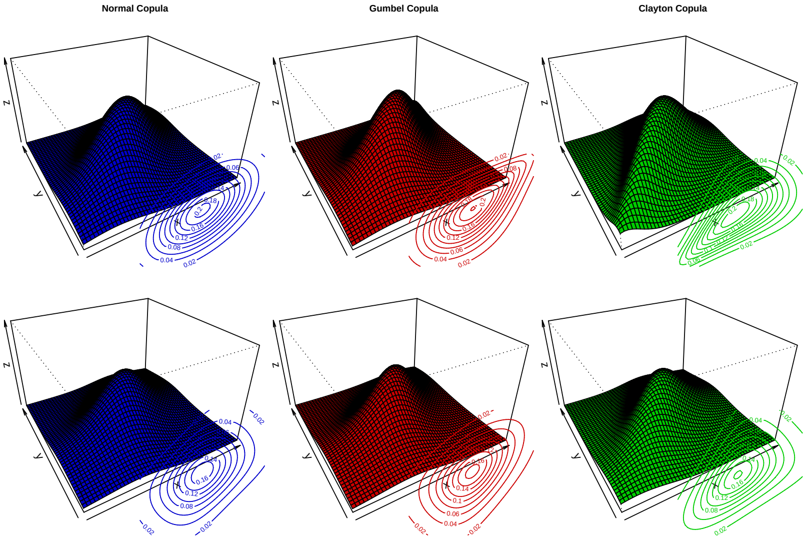
Bourdeau-Brien (2007) covers 871 publications

Copula Classes

1. elliptical
   - implied by well-known multivariate df's (Normal, t), derived through Sklar's theorem
   - do not have closed form expressions and are restricted to have radial symmetry
2. Archimedean

$$C(u_1, u_2) = \phi^{-1}\{\phi(u_1) + \phi(u_2)\}$$

   - allow for a great variety of dependence structures
   - closed form expressions
   - several useful methods for multivariate extension
   - not derived from mv df's using Sklar's theorem


Normal Copula
Gumbel Copula
Clayton Copula

Attractive Features

- It describes how the margins are tied together in the joint df
- the joint df is decomposed into the marginal dfs and a copula
- The marginal dfs and the copula can be modelled and estimated separately, independent of each other
- Given a copula, we can obtain many multivariate distributions by selecting different marginal dfs
- The copula is invariant under increasing and continuous transformations

CDO

- A static portfolio of 125 equally weighted CDS on European entities
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10)
- Maturities: 5Y, 7Y, 10Y

Construction

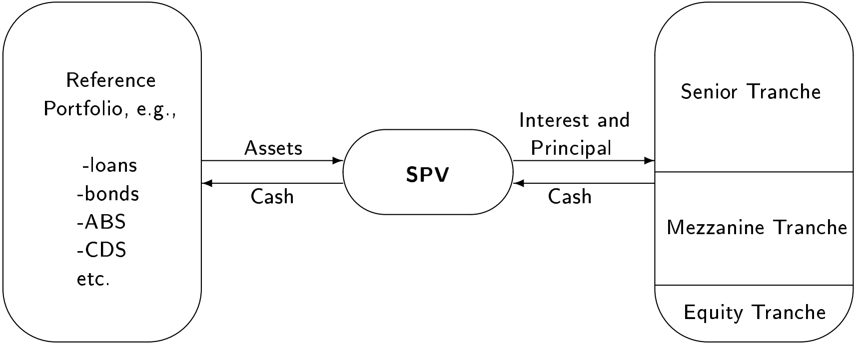


Figure 2: *CDO Transaction, Tranches*

Construction

		Attachment points (%)	
Tranche Number	Tranche Name	Lower l	Upper u
1	Equity	0	3
2	Mezzanine Junior	3	6
3	Mezzanine	6	9
4	Senior	9	12
5	Super Senior	12	22
6	Super Super Senior	22	100

Table 1: *Attachment points, iTraxx, CDO tranche structure*

CDO

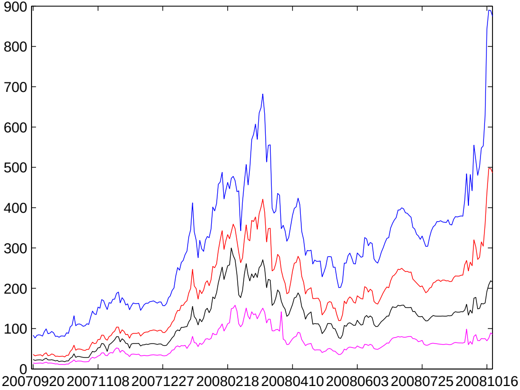


Figure 3: *Time series of iTraxx spreads, Series* 8, *Maturity:* 5 *years*, 21.03.2007 – 22.01.2008

Pricing

Standardized asset log-returns:

$$X_{i,t} = \sqrt{\rho_t}\, Y_t + \sqrt{1-\rho_t}\, Z_{i,t}$$

$\forall\ i = 1, \ldots, d$, where Y_t (systematic risk factor), $\{Z_{i,t}\}_{i=1}^{d}$ (idiosyncratic risk factors) are i.i.d. $\mathrm{N}(0,1)$. Hence:

$$(X_{1,t}, \ldots, X_{d,t})^\top \sim N(0, \Sigma_t),$$

with

$$\Sigma_t = \begin{pmatrix} 1 & \rho_t & \ldots & \rho_t \\ \rho_t & 1 & \ldots & \rho_t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_t & \rho_t & \ldots & 1 \end{pmatrix}$$

Gaussian ONE FACTOR model, constant ρ, iTraxx $d = 125$!!

Pricing

- Loss variable of ith firm until $t \in [0, T]$

$$\Gamma_{i,t} = I(\sqrt{\rho_t}\, Y_t + \sqrt{1-\rho_t}\, Z_{i,t} < C_t)$$

- Portfolio loss process

$$L_t = \frac{1-R}{d} \sum_{i=1}^{d} \Gamma_{i,t}$$

where R is the recovery rate equal for all credits in the portfolio.

Loss of the Tranche

The loss of the tranche j at time t

$$
\begin{aligned}
L_{j,t} &= \min\{\max(0, L_t - l_j); u_j - l_j\} \\
&= \begin{cases} 0, & L_t < l_j, \\ L_t - l_j, & l_j \le L_t \le u_j, \\ u_j - l_j, & L_t > u_j. \end{cases}
\end{aligned}
$$

Example Let j be the mezzanine tranche with the lower attachment point 6% and the upper attachment point 9%. Then

Loss of the portfolio	2	7	10
Loss of the tranche	0	1	3

CDO Premium

The premium s_j of tranche j is chosen in such a way that

1. fixed (premium) leg PL_j - the payments that tranche holders receive,
2. floating (protection) lef DL_j - the payments that tranche holders pay

are equal:

$$PL_j(\rho, s_j) = DL_j(\rho).$$

The premiums are constantly observed in the market!

Pricing

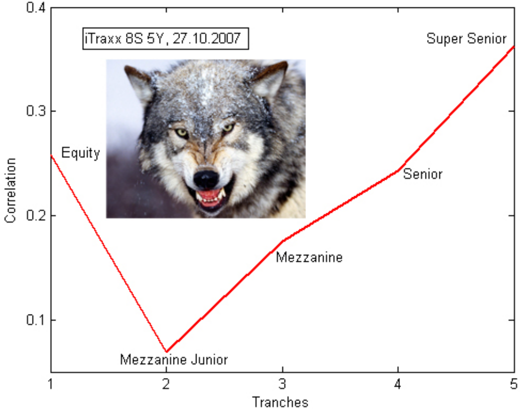


Figure 4: *Gaussian one factor model with constant correlation*

Gaussian Multi-Factor Model

Three factor model:

$$\Sigma = \begin{pmatrix}
\boxed{\begin{matrix}1&\dots&\rho_2\\ &\ddots& \\ \rho_2&\dots&1\end{matrix}} & \begin{matrix}\rho_1&\dots&\dots\\ \vdots&&\\ \rho_1&&\end{matrix} & & & \begin{matrix}\dots&\dots&\rho_1\\ &&\vdots\\ &&\vdots\end{matrix}\\
\begin{matrix}\rho_1&\dots&\rho_1\\ \vdots&&\\ \vdots&&\end{matrix} & \boxed{\begin{matrix}1&\dots&\rho_2\\ &\ddots& \\ \rho_2&\dots&1\end{matrix}} & & & \\
& & \ddots & & \\
& & & \boxed{\begin{matrix}1&\dots&\rho_3\\ &\ddots& \\ \rho_3&\dots&1\end{matrix}} & \begin{matrix}&&\vdots\\ &&\vdots\\ \rho_1&\dots&\rho_1\end{matrix}\\
\begin{matrix}\vdots&&\\ \vdots&&\\ \rho_1&\dots&\dots\end{matrix} & & & \begin{matrix}&&\rho_1\\ &&\vdots\\ \dots&\dots&\rho_1\end{matrix} & \boxed{\begin{matrix}1&\dots&\rho_3\\ &\ddots& \\ \rho_3&\dots&1\end{matrix}}
\end{pmatrix}$$

Qua de causa copula me placent: Statistics of joint events

Wolfgang K. Härdle
Ostap Okhrin

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Humboldt-Universität zu Berlin
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