Determining the Structure and Estimation of HAC

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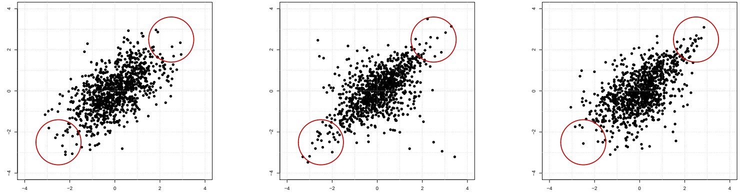
Introduction



Different distributions, however, the same margins and the same correlation coefficient !!!

Theorem (Sklar, 1959)

Let $X_1, \ldots, X_k$ *be random variables with marginal distribution functions* $F_1, \ldots, F_k$ *and joint distribution function* F. *Then there exists a* k*-dimensional copula* $C : [0,1]^k \rightarrow [0,1]$ *such that* $\forall x_1, \ldots, x_k \in \overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \ldots, x_k) = C(F_1(x_1), \ldots, F_k(x_k))$$

Archimedean Copula

Multivariate Archimedean copula $C : [0,1]^d \to [0,1]$ defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \qquad (1)$$

where $\phi : [0,\infty) \to [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.
Example 1

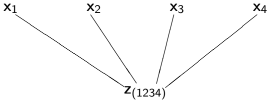
$$\begin{aligned} \phi_{Gumbel}(u,\theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \leq \theta < \infty \\ \phi_{Clayton}(u,\theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1,\infty)\backslash\{0\} \end{aligned}$$

Disadvantages: too restrictive, single parameter, exchangeable

Hierarchical Archimedean Copula

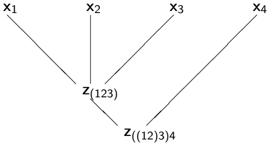
Simple AC with s=(1234)

$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$



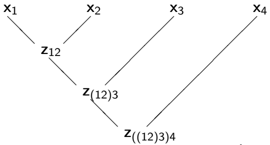
AC with s=((123)4)

$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$



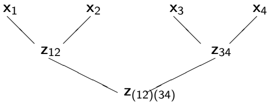
Fully nested AC with s=(((12)3)4)

$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$



Partially Nested AC with s=((12)(34))

$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$



$$C(u_1, u_2, u_3, u_4) = \phi(\phi^{-1}(\phi_{12}(\phi_{12}^{-1}(u_1) + \phi_{12}^{-1}(u_2))) + \phi^{-1}(\phi_{34}(\phi_{34}^{-1}(u_3) + \phi_{34}^{-1}(u_4))))$$

Hierarchical Archimedean Copula

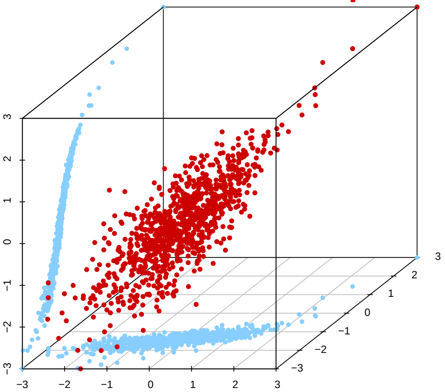


Abbildung 1: Scatterplot of the $C_{Gumbel}[C_{Gumbel}\{\Phi(x_1), t_2(x_2); \theta_1 = 2\}, \Phi(x_3); \theta_2 = 10]$

Hierarchical Archimedean Copula

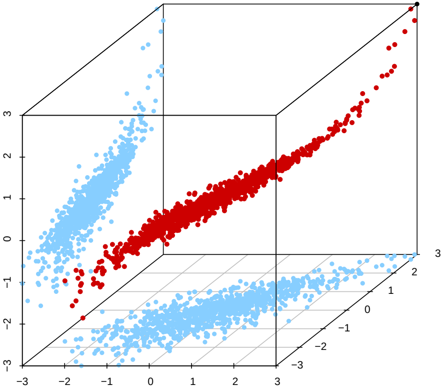


Abbildung 2: Scatterplot of the $C_{Gumbel}[\Phi(x_2), C_{Gumbel}\{t_2(x_1), \Phi(x_3); \theta_1 = 2\}; \theta_2 = 10]$

Hierarchical Archimedean Copula

Advantages of HAC:

- flexibility and wide range of dependencies:
  for $d = 10$ more than $2.8 \cdot 10^8$ structures
- dimension reduction:
  $d - 1$ parameters to be estimated
- subcopulas are also HAC

Determining Structure

$$
\begin{array}{r|}
(12) \rightsquigarrow \hat{\theta}_{12} \\
(13) \rightsquigarrow \hat{\theta}_{13} \\
(14) \rightsquigarrow \hat{\theta}_{14} \\
(23) \rightsquigarrow \hat{\theta}_{23} \\
(24) \rightsquigarrow \hat{\theta}_{24} \\
(34) \rightsquigarrow \hat{\theta}_{34} \\
\hline
(123) \rightsquigarrow \hat{\theta}_{123} \\
(124) \rightsquigarrow \hat{\theta}_{124} \\
(234) \rightsquigarrow \hat{\theta}_{234} \\
(134) \rightsquigarrow \hat{\theta}_{134} \\
(1234) \rightsquigarrow \hat{\theta}_{1234}
\end{array}
$$

Determining Structure

$$
\left.\begin{array}{r}
(12) \rightsquigarrow \hat{\theta}_{12} \\
(13) \rightsquigarrow \hat{\theta}_{13} \\
(14) \rightsquigarrow \hat{\theta}_{14} \\
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(34) \rightsquigarrow \hat{\theta}_{34} \\
\hline
(123) \rightsquigarrow \hat{\theta}_{123} \\
(124) \rightsquigarrow \hat{\theta}_{124} \\
(234) \rightsquigarrow \hat{\theta}_{234} \\
(134) \rightsquigarrow \hat{\theta}_{134} \\
(1234) \rightsquigarrow \hat{\theta}_{1234}
\end{array}\right| \text{best fit (13)} \rightsquigarrow
$$

$$z_{(13),i} = \hat{C}\{\hat{F}_1(x_{1i}), \hat{F}_3(x_{3i})\}$$

Determining Structure

$$
\begin{array}{c|}
(12) \rightsquigarrow \hat{\theta}_{12} \\
(13) \rightsquigarrow \hat{\theta}_{13} \\
(14) \rightsquigarrow \hat{\theta}_{14} \\
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(1234) \rightsquigarrow \hat{\theta}_{1234}
\end{array}
\quad \text{best fit (13)} \quad \rightsquigarrow \quad
\begin{array}{c|}
z_{(13),i} = \hat{C}\{\hat{F}_1(x_{1i}), \hat{F}_3(x_{3i})\} \\
\\
(13)2 \rightsquigarrow \hat{\theta}_{(13)2} \\
(13)4 \rightsquigarrow \hat{\theta}_{(13)4} \\
24 \rightsquigarrow \hat{\theta}_{24} \\
\hline
(13)24 \rightsquigarrow \hat{\theta}_{(13)24}
\end{array}
$$

Determining Structure

$$\left.\begin{array}{c} (12) \rightsquigarrow \hat{\theta}_{12} \\ (13) \rightsquigarrow \hat{\theta}_{13} \\ (14) \rightsquigarrow \hat{\theta}_{14} \\ (23) \rightsquigarrow \hat{\theta}_{23} \\ (24) \rightsquigarrow \hat{\theta}_{24} \\ (34) \rightsquigarrow \hat{\theta}_{34} \\ \hline (123) \rightsquigarrow \hat{\theta}_{123} \\ (124) \rightsquigarrow \hat{\theta}_{124} \\ (234) \rightsquigarrow \hat{\theta}_{234} \\ (134) \rightsquigarrow \hat{\theta}_{134} \\ (1234) \rightsquigarrow \hat{\theta}_{1234} \end{array}\right| \text{best fit (13)} \rightsquigarrow \left.\begin{array}{c} z_{(13),i} = \hat{C}\{\hat{F}_1(x_{1i}), \hat{F}_3(x_{3i})\} \\ \\ (13)2 \rightsquigarrow \hat{\theta}_{(13)2} \\ (13)4 \rightsquigarrow \hat{\theta}_{(13)4} \\ 24 \rightsquigarrow \hat{\theta}_{24} \\ \hline (13)24 \rightsquigarrow \hat{\theta}_{(13)24} \end{array}\right| \text{best fit ((13)4)} \rightsquigarrow z_{((13)4),i} = \hat{C}\{z_{(13)i}, \hat{F}_4(x_{4i})\}$$

Determining Structure

$$
\left.\begin{array}{r}
(12) \rightsquigarrow \hat{\theta}_{12} \\
(13) \rightsquigarrow \hat{\theta}_{13} \\
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(24) \rightsquigarrow \hat{\theta}_{24} \\
(34) \rightsquigarrow \hat{\theta}_{34} \\ \hline
(123) \rightsquigarrow \hat{\theta}_{123} \\
(124) \rightsquigarrow \hat{\theta}_{124} \\
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(134) \rightsquigarrow \hat{\theta}_{134} \\
(1234) \rightsquigarrow \hat{\theta}_{1234}
\end{array}\right| \text{best fit } (13) \rightsquigarrow
\left.\begin{array}{c}
z_{(13),i} = \hat{C}\{\hat{F}_1(x_{1i}), \hat{F}_3(x_{3i})\} \\
\\
(13)2 \rightsquigarrow \hat{\theta}_{(13)2} \\
(13)4 \rightsquigarrow \hat{\theta}_{(13)4} \\
24 \rightsquigarrow \hat{\theta}_{24} \\ \hline
(13)24 \rightsquigarrow \hat{\theta}_{(13)24}
\end{array}\right| \text{best fit } ((13)4) \rightsquigarrow
\begin{array}{c}
z_{((13)4),i} = \hat{C}\{z_{(13)i}, \hat{F}_4(x_{4i})\} \\
\\
((13)4)2 \rightsquigarrow \hat{\theta}_{((13)4)2}
\end{array}
$$

Determining Structure

$$
\begin{array}{r|c}
(12) \rightsquigarrow \hat{\theta}_{12} & \\
(13) \rightsquigarrow \hat{\theta}_{13} & \\
(14) \rightsquigarrow \hat{\theta}_{14} & \\
(23) \rightsquigarrow \hat{\theta}_{23} & \\
(24) \rightsquigarrow \hat{\theta}_{24} & \\
(34) \rightsquigarrow \hat{\theta}_{34} & \text{best fit (13)} \\
\hline
(123) \rightsquigarrow \hat{\theta}_{123} & \\
(124) \rightsquigarrow \hat{\theta}_{124} & \\
(234) \rightsquigarrow \hat{\theta}_{234} & \\
(134) \rightsquigarrow \hat{\theta}_{134} & \\
(1234) \rightsquigarrow \hat{\theta}_{1234} &
\end{array}
\rightsquigarrow
\begin{array}{c|c}
z_{(13),i} = \hat{C}\{\hat{F}_1(x_{1i}), \hat{F}_3(x_{3i})\} & \\
& \\
(13)2 \rightsquigarrow \hat{\theta}_{(13)2} & \\
(13)4 \rightsquigarrow \hat{\theta}_{(13)4} & \text{best fit ((13)4)} \\
24 \rightsquigarrow \hat{\theta}_{24} & \\
\hline
(13)24 \rightsquigarrow \hat{\theta}_{(13)24} &
\end{array}
\rightsquigarrow
\begin{array}{c}
z_{((13)4),i} = \hat{C}\{z_{(13)i}, \hat{F}_4(x_{4i})\} \\
\\
((13)4)2 \rightsquigarrow \hat{\theta}_{((13)4)2}
\end{array}
$$

Estimation: multistage MLE with nonparametric and parametric margins

Criteria for grouping: goodness-of-fit tests, parameter-based method, etc.

Criteria for grouping: I

We need a criterion for grouping variables at each level of the copula.
Alternatives:

- goodness-of-fit tests
  - dimension dependent
  - KS type tests are difficult to implement
  - possible choice $\rightsquigarrow$ Chen et al. (2004, WP of LSE), Fermanian (2005, JMA)
- distance measures
  - dimension dependent
- parameter-based methods

  Note that, if the true structure is (123) then

  $$\theta_{(12)} = \theta_{(13)} = \theta_{(23)} \quad = \theta_{(123)}$$

  - heuristic methods
  - test-based methods
- tests on exchangeability

Criteria for grouping: II

To guarantee that C is a HAC we assume that $\phi_{k-i}^{-1} \circ \phi_{k-j} \in \mathcal{L}^*$, $i < j$ with

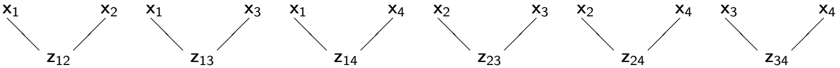
$$\mathcal{L}^* = \{\omega : [0,\infty) \rightarrow [0,\infty) \,|\, \omega(0) = 0,\ \omega(\infty) = \infty,\ (-1)^{j-1}\omega^{(j)} \geq 0,\ j \geq 1\}.$$

for most of the generator functions the parameters should decrease from the lowest level to the highest

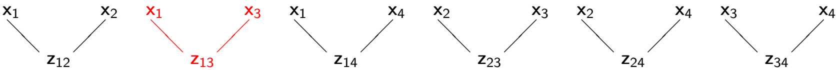
Theorem

Let F be an arbitrary multivariate distribution function based on HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.

Criteria for grouping based on θ's

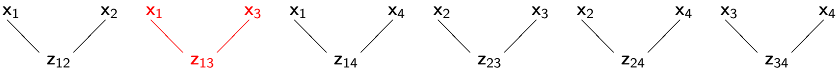


Criteria for grouping based on θ's



$$\max\{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{34}\} = \hat{\theta}_{13} \quad \Rightarrow$$

Criteria for grouping based on θ's



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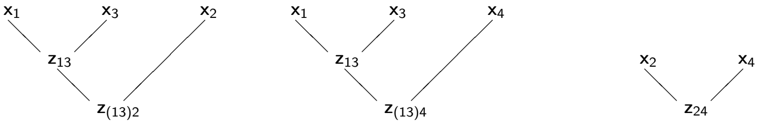
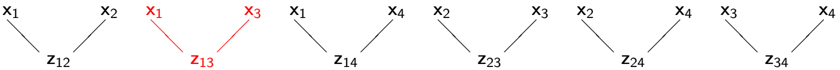


Criteria for grouping based on θ's



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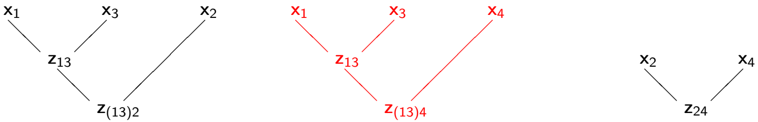
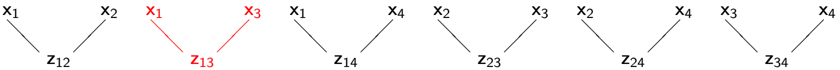


$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4} \quad \Rightarrow$$

Criteria for grouping based on θ's



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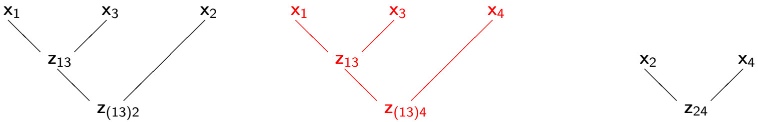

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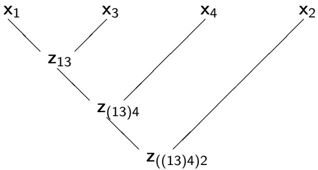
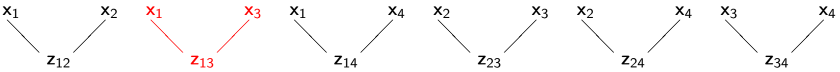
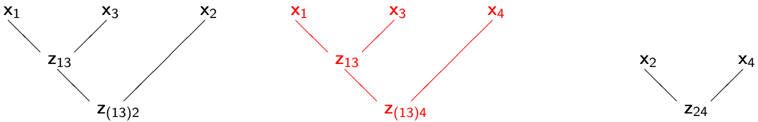


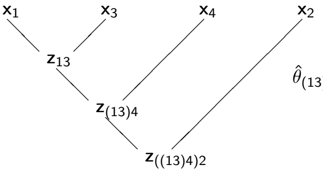
Criteria for grouping based on θ's



$$\max\{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{34}\} = \hat{\theta}_{13} \quad \Rightarrow$$

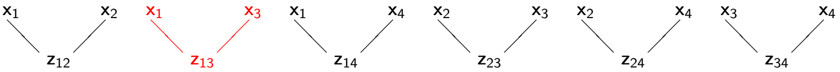


$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4} \quad \Rightarrow$$



$$\hat{\theta}_{(13)} \approx \hat{\theta}_{((13)4)} \Rightarrow$$

Criteria for grouping based on θ's



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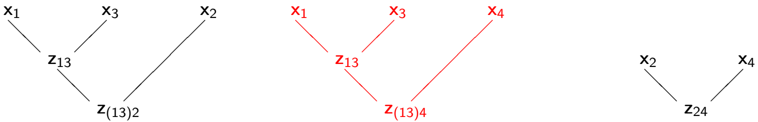

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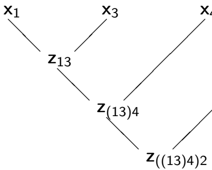
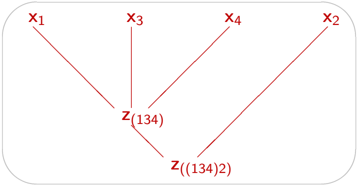


$$\hat{\theta}_{(13)} \approx \hat{\theta}_{((13)4)} \Rightarrow$$



Estimation Issues (parametric margins)

Multistage MLE

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\alpha}_1'}, \ldots, \frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\alpha}_k'}, \frac{\partial \mathcal{L}_{k+1}}{\partial \boldsymbol{\theta}_1'}, \ldots, \frac{\partial \mathcal{L}_{k+p}}{\partial \boldsymbol{\theta}_p'}\right)' = \mathbf{0},$$

$$\begin{aligned}
\text{where} \quad \mathcal{L}_j &= \sum_{i=1}^{n} l_j(\mathbf{X}_i), \ \text{for } j = 1, \ldots, k+p, \\
l_j(\mathbf{X}_i) &= log\, f_j(x_{ji}, \boldsymbol{\alpha}_j), \ \text{for } j = 1, \ldots, k, i = 1, \ldots, n, \\
l_{j+k}(\mathbf{X}_i) &= log\left[c(\{\phi_\ell, \boldsymbol{\theta}_\ell\}_{\ell=1,\ldots,p};\, s_j)(\{F_m(x_{mi}, \boldsymbol{\alpha}_m)\}_{m \in s_j})\right. \\
&\times \left.\prod_{m \in s_j} f_m(x_{mi}, \boldsymbol{\alpha}_m)\right] \quad \text{for } j = 1, \ldots, p, \ i = 1, \ldots, n
\end{aligned}$$

Theorem

Let each marginal distribution f_r for $r = 1, \ldots, k$ and the copula density at each level of the hierarchy satisfy the regularity conditions of Theorem 5.2.2 of Sen and Singer (1993). Then

$$n^{\frac{1}{2}}(\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}) \overset{a}{\sim} N(0, \mathbf{B}^{-1} \circ \mathbf{B}^{-1}),$$

Estimation Issues (nonparametric margins)

$$\hat{F}_j(x) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{I}(x_{ji} \le x), \quad j = 1, \ldots, k.$$

Canonical MLE

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\theta}_1'}, \ldots, \frac{\partial \mathcal{L}_p}{\partial \boldsymbol{\theta}_p'}\right)' = \mathbf{0},$$

where

$$\mathcal{L}_j = \sum_{i=1}^{n} l_j(\mathbf{X}_i) \text{ for } j = 1, \ldots, p.$$

$$l_j(\mathbf{X}_i) = \log\Big[c(\{\phi_\ell, \boldsymbol{\theta}_\ell\}_{\ell=1,\ldots,j};\, s_j)(\{\hat{F}_m(x_{mi})\}_{m\in s_j}) \times \prod_{m\in s_j} \hat{f}_m(x_{mi})\Big] \quad \text{for } j = 1, \ldots, p.$$

Theorem

Under regularity conditions, estimator $\hat{\boldsymbol{\theta}}$ is consistent and

$$n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{a}{\sim} N(\mathbf{0}, \mathbf{B}^{-1} \circ \mathbf{B}^{-1}),$$

Data and Copula

- daily returns of Commerzbank (CBK), Merck (MRK), ThyssenKrupp (TKA) and Volkswagen (VOW)

- timespan = [13.11.1998 - 18.10.2007] ($n = 2400$)

- $\mathcal{M} = \{\phi = \exp(-u^{1/\theta})\}$ - Gumbel generator
- GARCH-residuals are conditionally distributed with estimated copula

$$\varepsilon \sim C\{F_1(x_1), \ldots, F_d(x_d); \theta_t\}$$

  where $F_1, \ldots, F_d$ are marginal distributions and θ_t are the copula parameters.

Fit in time: I

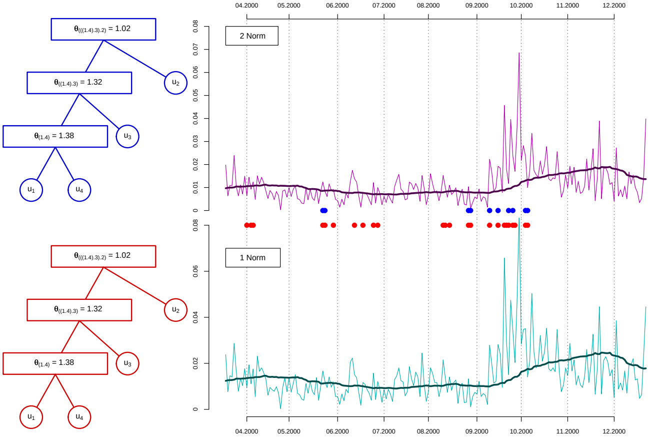


Abbildung 3: Film of time-varying HAC

Fit in time: II

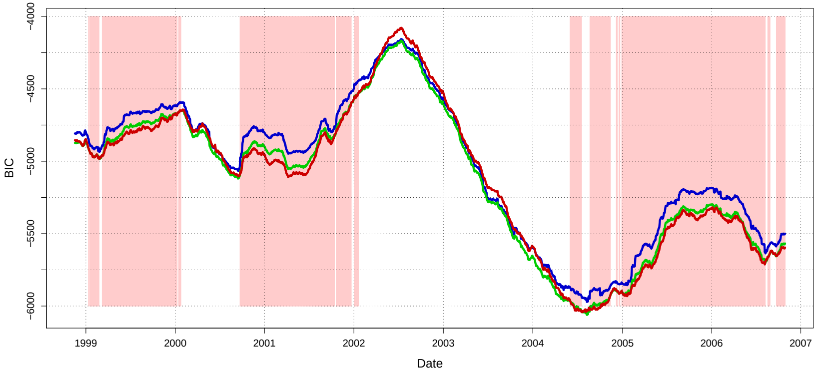


Abbildung 4: Time-varying HAC: BIC for the multivariate t distribution, multivariate $\mathcal{N}$ distribution and estimated HAC. Shaded areas represents intervals where HAC-based distribution outperforms t and $\mathcal{N}$

Fit in time: III

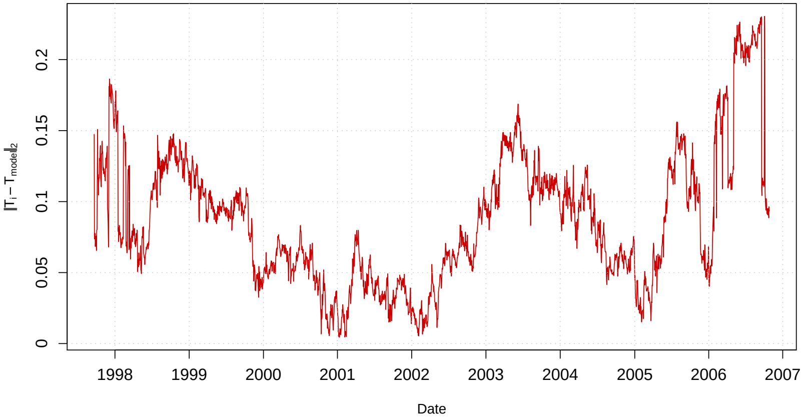


Abbildung 5: Norm of the difference between Kendall's matrices for the model and from the data

Abbildung 6: Model fit for the true structure (123)(45)

Averages of the Kullback-Leibler Divergence

method	copula structure	Chen stat	$\mathcal{K}$ ($\hat{\mu}_{\mathcal{K}}$)	$\hat{\sigma}^2_{\mathcal{K}}$
$\tau_{\triangle\tau>0}$	$((1.2.3)_{3.946}.(4.5)_{3.168})_{2.079}$	1.52258	-0.00279 (-0.00277)	0.00001
τ_{binary}	$(((1.3)_{4.202}.2)_{4.15}.(4.5)_{3.09})_{2.131}$	3.04989	-0.00296 (-0.00295)	0.00001
$\mathcal{N}$ with $\mathcal{N}$ margins			1.07443 (1.07364)	0.00413
$\mathcal{N}$ with $\mathcal{U}$ margins			0.28163 (0.28273)	0.00082
t with t margins			1.10359 (1.10362)	0.00323
t with $\mathcal{U}$ margins			0.19891 (0.19909)	0.00050
sAC	$(1.2.3.4.5)$	85.53546	0.81056 (0.80909)	0.00319
CHEN	$((1.3)_{4.517}.(2.4.5)_{2.341})_{2.34}$	31.43151	0.61117 (0.61318)	0.07814
θ	$((1.3.4.5)_{2.288}.2)_{2.286}$	82.51007	0.69741 (0.56016)	0.14222
θ_{binary}	$(((1.(2.3)_{4.39})_{4.282}.5)_{2.078}.4)_{2.077}$	3.92928	0.13221 (0.13256)	0.00038
$\theta_{binary\ aggr.}$	$(((1.3)_{4.26}.2)_{3.868}.(4.5)_{3.093})_{2.259}$	2.73659	0.02164 (0.02139)	0.00018

Averages of the Chen Statistics

method	copula structure	Chen stat ($\hat{\mu}_{Chen\ stat}$)	$\mathcal{K}$	$\hat{\sigma}^2_{Chen\ stat}$
$\tau_{\triangle\tau>0}$	$((1.2.3)_{4.125}.(4.5)_{3.054})_{2.085}$	1.86098 (1.85453)	-0.00089	1.42871
τ_{binary}	$(((1.3)_{4.184}.2)_{4.104}.(4.5)_{3.054})_{2.085}$	1.76319 (1.77912)	-0.00175	1.32981
sAC	$(1.2.3.4.5)$	88.84195 (88.85004)	0.70363	68.12699
CHEN	$((1.2)_{4.316}.(3.4.5)_{2.256})_{2.255}$	31.55789 (32.41887)	0.58451	490.05904
θ	$((1.2.4.5)_{2.376}.3)_{2.375}$	56.07732 (56.90958)	0.76863	1407.63193
θ_{binary}	$((((1.2)_{4.487}.3)_{4.469}.5)_{2.247}.4)_{2.246}$	4.78944 (4.82685)	0.11193	4.38843
$\theta_{binary\ aggr.}$	$(((1.3)_{4.228}.2)_{3.68}.(4.5)_{3.369})_{2.333}$	2.25313 (2.24800)	0.02068	1.91354

Abbildung 7: Model fit for the true structure (12(34))5

Averages of the Kullback-Leibler Divergence				
method	**copula structure**	**Chen stat**	$\mathcal{K}$ ($\hat{\mu}_{\mathcal{K}}$)	$\hat{\sigma}^2_{\mathcal{K}}$
$\tau_{\triangle\tau>0}$	$(((1.2.4)_{3.275}.3)_{3.272}.5)_{2.103}$	3.488708	0.06119 (0.06181)	0.00123
τ_{binary}	$((1.(2.(3.4)_{4.039})_{3.087})_{3.085}.5)_{1.961}$	1.29107	-0.00261 (-0.00259)	0.00001
$\mathcal{N}$ with $\mathcal{N}$ margins			1.08846 (1.08894)	0.00352
$\mathcal{N}$ with $\mathcal{U}$ margins			0.28895 (0.28947)	0.00081
t with t margins			1.11346 (1.11360)	0.00289
t with $\mathcal{U}$ margins			0.20231 (0.20192)	0.00055
sAC	(1.2.3.4.5)	78.60395	0.50203 (0.50226)	0.00208
CHEN	$(((1.2)_{3.22}.3)_{3.177}.(4.5)_{2.116})_{2.114}$	8.54422	0.30505 (0.30360)	0.02316
θ	$(((1.2.3)_{3.207}.4)_{3.205}.5)_{2.15}$	5.74121	0.07867 (0.07856)	0.00022
θ_{binary}	$(((1.(3.4)_{4.157})_{3.099}.2)_{3.012}.5)_{2.028}$	2.29313	0.00263 (-0.00265)	0.00001
$\theta_{binary\ aggr.}$	$(((3.4)_{4.32}.1.2)_{3.268}.5)_{1.83}$	1.22026	0.01939 (0.01930)	0.00009
Averages of the Chen Statistics				
method	**copula structure**	**Chen stat** ($\hat{\mu}_{Chen\ stat}$)	$\mathcal{K}$	$\hat{\sigma}^2_{Chen\ stat}$
$\tau_{\triangle\tau>0}$	$(((1.2.4)_{3.12}.3)_{3.118}.5)_{2.091}$	3.81102 (3.81492)	0.06809	6.82397
τ_{binary}	$(((1.(3.4)_{4.122})_{3.155}.2)_{3.07}.5)_{2.027}$	1.93392 (1.95050)	0.00043	1.49485
sAC	(1.2.3.4.5)	86.24487 (86.27755)	0.47955	142.71385
CHEN	$(((1.3)_{2.835}.5)_{1.987}.(2.4)_{2.898})_{1.986}$	16.26255 (16.51243)	0.45255	281.61467
θ	$(((1.2.4)_{3.009}.3)_{3.007}.5)_{1.973}$	4.23520 (4.22160)	0.08314	6.22898
θ_{binary}	$(((1.(3.4)_{4.122})_{3.155}.2)_{3.07}.5)_{2.027}$	1.93392 (1.95482)	0.00043	1.51995
$\theta_{binary\ aggr.}$	$(((3.4)_{4.195}.1.2)_{3.305}.5)_{1.724}$	2.56085 (2.52641)	0.01433	3.28738