

Asymptotic behavior of S -stopped branching processes

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Outline

1. Definitions
2. First Main Theorem
3. Assumptions
4. Main Results
5. Further Research
6. References

Definition I

- (X, \mathcal{A}) – measured state space with \mathcal{A} – σ -algebra on X
- $P(t, x, A)$ – transition probability, with t – time, $x \in X$ and $A \in \mathcal{A}$
- time is **discrete**
- T_1, \dots, T_n, \dots – types of particles
- only **one** particle at the point $x \in X$ at time point $t = 0$
- $\mu_{xt}(A)$ – a random measure
- $\mathbb{N}_0 = \{0, 1, 2, \dots\}$
- $\mathbb{N}_0^\infty = \mathbb{N}_0 \times \mathbb{N}_0 \times \dots$
- $\mathcal{E}(i)$ – the particle of type i

Definition II

Measure which describe the number of particles of each type at time point t .

Multivariate measure μ is based on μ

$$\mu_{\mathbf{x}t}(A) = \left\{ \begin{array}{ll} \sum_{i=0}^{\infty} \sum_{j=1}^{n_i} \mu_{x_{ij}t}(x_m), & \text{if } x_m \in A \\ 0, & \text{else} \end{array} \right\}_{m=0}^{\infty},$$

where $\mathbf{x} = \{x_{11}, \dots, x_{1n_1}, x_{21}, \dots, x_{2n_2}, \dots\}$ and $x_{ij} \in X$ is the j -th element of the i -th type.

Definition III

Definition

Let us fix countable subset $S \subset \mathbb{N}_0^\infty$, $0 \notin S$. *Stopped*, or *S-stopped* branching process is the process $\xi_{xt}(X)$, defined for $t = 1, 2, \dots$ and $x \in \mathbb{N}_0^\infty$ by equalities

$$\xi_{xt}(X) = \begin{cases} \mu_{xt}(X), & \text{if } \forall v, 0 \leq v < t, \mu_{xv}(X) \notin S \\ \mu_{xu}(X), & \text{if } \forall v, 0 \leq v < u, \mu_{xv}(X) \notin S, \\ & \mu_{xu}(X) \in S, u < t \end{cases}$$

Definition IV

Definition

Let

$$q_r^n(t) = P\{\xi_{nt}(X) = r\}$$

be the probability of *extinction* of the S -stopped branching process $\xi_{xt}(X)$ into state $r \in S$ till time t , which starts from state $n \in \mathbb{N}_0^\infty$ and $q_r^n = \lim_{t \rightarrow \infty} q_r^n(t)$.

Definition V

$$\begin{aligned}\widehat{P}(t, \mathbf{x}, \mathbf{y}) &= P\{\mu_{\mathbf{x}t}(X) = \mathbf{y}\}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{N}_0^\infty \\ \widetilde{P}(t, \alpha, \mathbf{r}) &= \begin{cases} \widehat{P}(1, \alpha, \mathbf{r}), & t = 1; \\ \sum_{\beta \notin S} \widehat{P}(1, \alpha, \beta) \widetilde{P}(t-1, \beta, \mathbf{r}), & t \geq 2. \end{cases}\end{aligned}$$

where $\alpha \notin S$, $\alpha \neq 0$, $r \in S$, \widetilde{P} is the conditional probability of the event

$$\{\mu_{\alpha t}(X) = \mathbf{r}\} \bigcap \bigcap_{l'=1}^{t-1} \{\mu_{\alpha l'}(X) \notin S\}.$$

generated functional

$$h(t, s(\cdot)) = \mathbb{E} \exp \left\{ \int_X \ln s(z) \mu_t(dz) \right\}, \quad t \in \mathbb{N}$$

where $s(\cdot)$ is an measurable bounded function

$$h(t + \tau, s) = h(t, h(\tau, s))$$

$$h(s) = h(1, s)$$

Asymptotic behavior of S -stopped branching processes



First Main Theorem

Theorem

$\forall \mathbf{n} \notin S, \mathbf{n} \neq 0, \mathbf{r} \in S, t \geq 1$ holds

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{\alpha \in S} \sum_{l=1}^t c_{\alpha \mathbf{r}}(t, l) \hat{P}(l, \mathbf{n}, \alpha),$$

$$\text{where } c_{\alpha \mathbf{r}}(t+1, l+1) = c_{\alpha \mathbf{r}}(t, l),$$

$$c_{\alpha \mathbf{r}}(t+1, 1) = \delta_{\alpha \mathbf{r}} - \sum_{l=1}^{t-1} \tilde{P}(l, \alpha, \mathbf{r})$$

$$c_{\alpha \mathbf{r}}(1, 1) = \delta_{\alpha \mathbf{r}}.$$

sketch of the proof.

based on the moment of the first attendance in S , the recursive use of the definitions of both transitions probabilities and that

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{l=1}^t \tilde{P}(l, \mathbf{n}, \mathbf{r})$$

Asymptotic behavior of S -stopped branching processes



Assumptions

Assumption (I)

The process is indecomposable, noncyclic and subcritical.

Assumption (II)

$\forall i, j = 1, 2, \dots$ $E\{\mu_{\mathcal{E}(j)1}(x_i) \log \mu_{\mathcal{E}(j)1}(x_i)\}$ is finite, for $x_i \in X$.

Assumption (III)

If $h_{ij}(s) = \frac{\partial h_i(s)}{\partial s_j}$, then for all j , $1 \leq j < \infty$ there exists such i , $1 \leq i < \infty$, that $h_{ij}(0)$ are positive.

Main Result I ($t \rightarrow \infty$)

Lemma

Under Assumptions [I], [II] and [III]

$\lim_{t \rightarrow \infty} P\{\mu_{nt}(X) = \mathcal{E}(i) | \mathbf{n} \neq 0\} = p_{\mathcal{E}(i)}^* > 0$, for all $i = 1, 2, \dots$

sketch of the proof.

based on the existence of the limit $\lim_{t \rightarrow \infty} P\{\mu_{nt}(X) = \mathbf{k} | \mathbf{n} \neq 0\} = p_{\mathbf{k}}^*$

and that generating function $h^*(s) = \sum_{\mathbf{k} \in \mathcal{N}_0^\infty} p_{\mathbf{k}}^* s^{\mathbf{k}}$ fulfills

$1 - h^*(h(\cdot)) = \delta(1 - h^*(s))$ and fulfills the main differential equation. \square

Theorem

If $c_{\alpha\mathbf{r}} = \lim_{t \rightarrow \infty} c_{\alpha\mathbf{r}}(t, l) = \delta_{\alpha\mathbf{r}} - \sum_{u=1}^{\infty} \tilde{P}(u, \alpha\mathbf{r})$ then under the Assumption

[I] $q_{\mathbf{r}}^{\mathbf{n}} = \sum_{l=1}^{\infty} \sum_{\alpha \in S} c_{\alpha\mathbf{r}} \hat{P}(l, \mathbf{n}, \alpha)$, $\forall \mathbf{n} \notin S, \mathbf{r} \in S$

Main Result II ($t \rightarrow \infty$ and $\bar{n} \rightarrow \infty$)

Theorem

Let Assumptions [I],[II] are fulfilled and $\lim_{\bar{n} \rightarrow \infty} (n_i / \bar{n}) = a_i$, where $a = (a_1, a_2, \dots)$. In this case for $\mathbf{r} \in S$, $\mathbf{n} \notin S$, $\mathbf{n} \neq 0$

$$q_{\mathbf{r}}^{\mathbf{n}} - H(\log_{\delta} \bar{n}) \rightarrow 0, \text{ for } \bar{n} \rightarrow \infty.$$

$H(x)$ is a periodic function with period 1






Lemma

Under Assumptions (1), (2), there exists such constant $\Theta > 0$, that for some number n_0 holds $q_{\mathbf{r}}^{\mathbf{n}} > \Theta$, for $\forall \mathbf{n}$ with $\bar{n} \geq n_0$ and $\forall \mathbf{r} \in S$.

Further Research

- Uncountable number of types of particles
- Immigration influence on S -stopped branching processes
- Critical extension
- Other properties

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