Hierarchical Archimedean Copulae

Ostap Okhrin
ostap.okhrin@wiwi.hu-berlin.de

Ladislaus von Bortkiewicz
Chair of Statistics
Humboldt–Universität zu Berlin

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09

In the mid-'80s, *Wall Street* turned to the quants – *brainy financial engineers* – to invent new ways to boost profits.

Their methods for minting money worked brilliantly...

until one of the them devastated the global economy.

Here's what killed your 401(k). *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick and fatally flawed way to assess risk.*

$$\Pr[T_A < 1, T_B < 1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Gamma - The all-powerful correlation parameter, which reduces correlation to a single constant-something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.

Example

- we pay 200 EUR for the chance to win 1000 EUR, if DAX returns decrease by 2%

$$\mathrm{P}_{DAX}(r_{DAX} \leq -0.02) = F_{DAX}(-0.02) = 0.2$$

- we pay 200 EUR for the chance to win 1000 EUR, if DJ returns decrease by 1%

$$\mathrm{P}_{DJ}(r_{DJ} \leq -0.01) = F_{DJ}(-0.01) = 0.2$$

Example

- we get 1000 EUR if DAX and DJ indices decrease simultaneously by 2% and 1% respectively.
  how much are we ready to pay in this case?

$$
\begin{aligned}
\mathrm{P}\{(r_{DAX} \leq -0.02) \wedge (r_{DJ} \leq -0.01)\}
&= F_{DAX,DJ}(-0.02, -0.01) \\
&= C\{F_{DAX}(-0.02), F_{DJ}(-0.01)\} \\
&= C(0.2, 0.2).
\end{aligned}
$$

Outline

Univariate Case

Let $x_1, \dots, x_n$ be realizations of the random variable X
$X \sim F$, where F is unknown

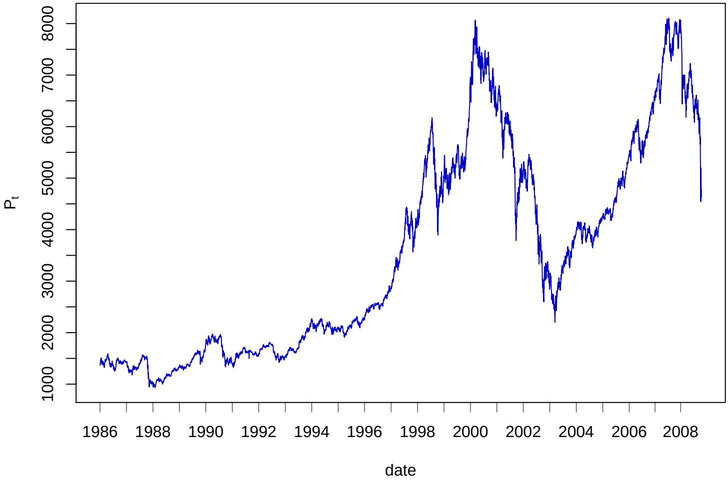
Example

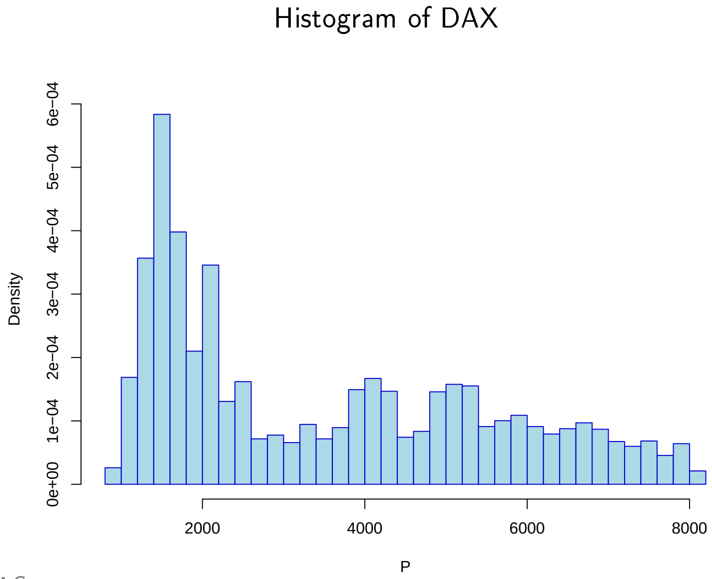
- x_i are returns of the asset for one firm at the day t_i
- x_i are numbers of sold albums *The Man Who Sold the World* by *David Bowie* at day t_i

What is a good approximation of F ?

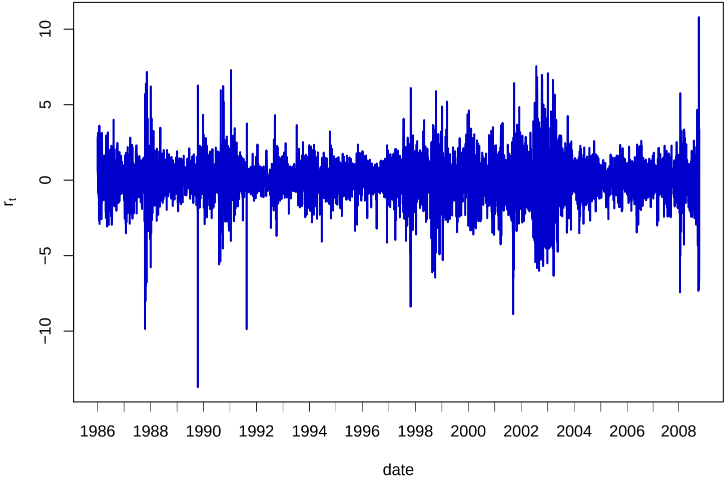
traditional or modern approach

DAX (P_t)

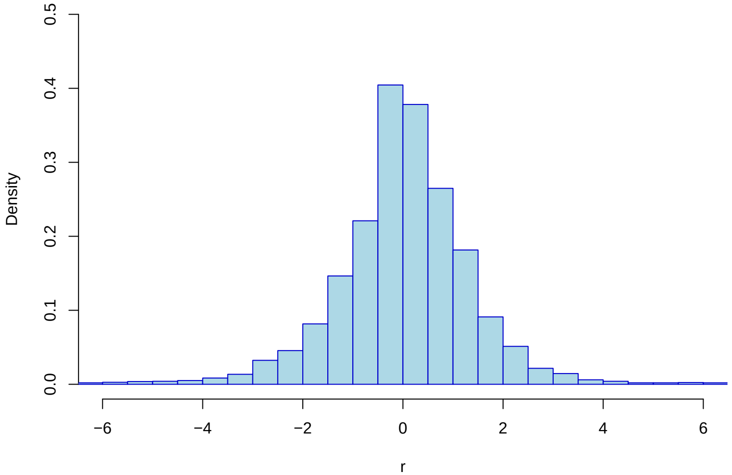

8000
7000
6000
5000
4000
3000
2000
1000
P_t
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
date


Histogram of DAX
Density
0e+00
1e−04
2e−04
3e−04
4e−04
5e−04
6e−04
2000
4000
6000
8000
P

DAX returns ($r_t = \log \frac{P_t}{P_{t-1}}$)



Histogram of DAX returns



Traditional approach:

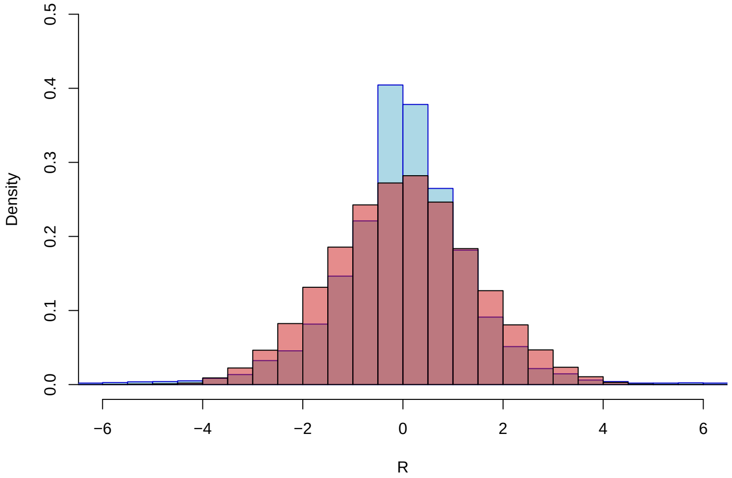
F_0 – known distribution

- parameters of F_0 are estimated from the sample $x_1, \dots, x_n$
  - $F_0 = N(\mu, \sigma^2) \;\Rightarrow\; (\mu, \sigma)$, here $\widehat{\mu} = \overline{x}$, $\widehat{\sigma}^2 = \widehat{s}^2$
  - $F_0 = St(\alpha, \beta, \mu, \sigma^2) \;\Rightarrow\; (\alpha, \beta, \mu, \sigma)$ are estimated by Hull Estimator, Tail Exponent Estimation, etc.
- check the appropriateness of F_0 by a test (KS type)

$$H_0 : F = F_0 \quad \text{vs} \quad H_1 : F \neq F_0$$

- if test confirm F_0, use $\widehat{F}_0$

Fit of the Normal distribution to DAX returns
($\widehat{\mu} = 0.0002113130,\ \widehat{\sigma}^2 = 0.0002001865$)



Modern approach: calculate the edf

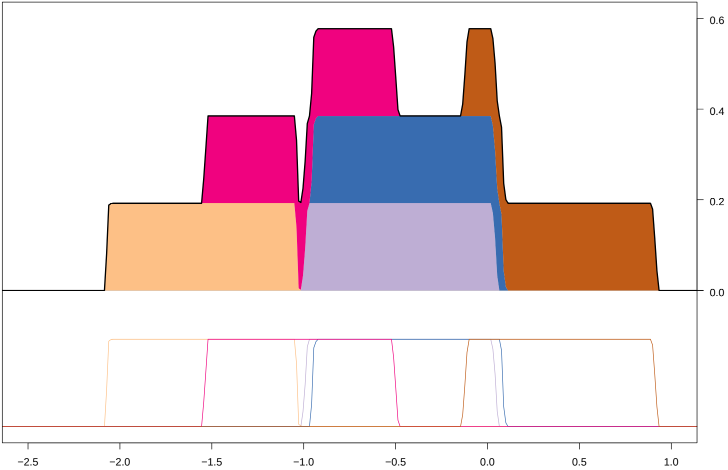
$$\widehat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}\{X_i \leq x\},$$

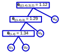
or the nonparametric kernel smoother

$$\widehat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

name	$K(u)$
Uniform	$\frac{1}{2}\mathbf{I}\{\lvert u\rvert \leq 1\}$
Epanechnikov	$\frac{3}{4}(1 - u^2)\mathbf{I}\{\lvert u\rvert \leq 1\}$
Gaussian	$\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\}$

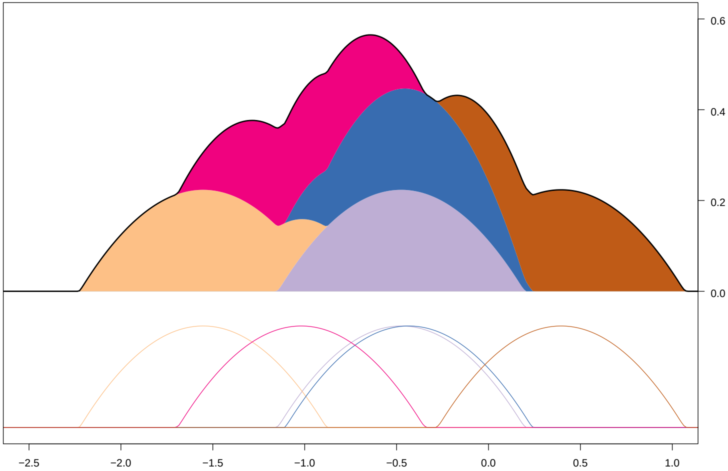
Kernel smoothing with UNI kernel
$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$





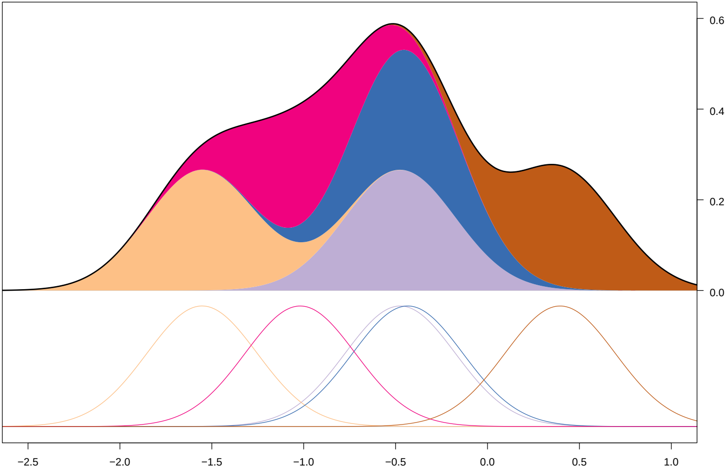
Kernel smoothing with EPA kernel

$$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$$

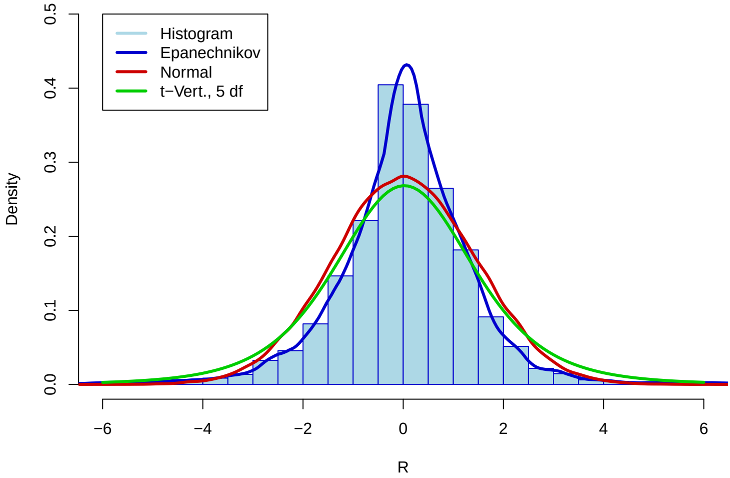


Kernel smoothing with GAU kernel

$$x = (-0.475, -1.553, -0.434, -1.019, 0.395)$$



The estimated density of DAX returns



Multivariate Case

$\{x_{1i}, \ldots, x_{di}\}_{i=1,\ldots,n}$ is the realization of the vector $(X_1, \ldots, X_d) \sim \mathbf{F}$, where $\mathbf{F}$ is unknown.

Example

- $\{x_{1i}, \ldots, x_{di}\}_{i=1,\ldots,n}$ are returns of the d assets in the portfolio at day t_i
- $(x_{1i}, x_{2i})^\top$ are numbers of sold albums *The Man Who Sold The World* by *David Bowie* and singles *I Saved The World Today* by *Eurythmics* at day t_i

Multivariate Case

What is a good approximation of F ?

`traditional` or `modern` approach

Very flexible approximation to **F** is challenging in high dimension due to curse of dimensionality.

Traditional approach: Mainly restricted to the class of elliptical distributions: Normal or t distributions

$$f_N(x_1, \dots, x_d) = \frac{1}{\sqrt{|\Sigma|(2\pi)^d}} \exp\left\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}$$

Drawbacks of the elliptical distributions:

1. does not often describe financial data properly
2. huge number of parameters to be estimated

   f.e. for Normal distribution: $\underbrace{\frac{d(d-1)}{2}}_{in\ dependency} + \underbrace{2d}_{in\ margins}$

3. ellipticity

Simulate $X \sim \boldsymbol{N}(\mu, \Sigma)$ with the sample size $n = 1000$ and estimate the parameters $(\widehat{\mu}, \widehat{\Sigma})$

$$\Sigma = \begin{pmatrix} 1.5 & 0.7 & 0.2 \\ 0.7 & 1.3 & -0.4 \\ 0.2 & -0.4 & 0.3 \end{pmatrix} \Rightarrow \widehat{\Sigma} = \begin{pmatrix} 1.461 & 0.726 & 0.181 \\ 0.726 & 1.335 & -0.408 \\ 0.181 & -0.408 & 0.301 \end{pmatrix}$$

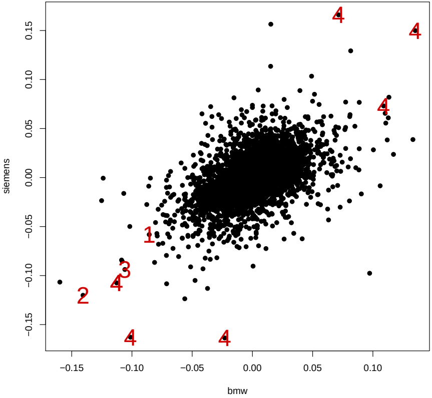
$$\mu = (0,\ 0,\ 0) \Rightarrow \widehat{\mu} = (0.0175,\ -0.0022,\ 0.0055)$$

$\widehat{\Sigma}$ and Σ are not close to each other for only 3 dimensions and quiet big sample

"Extreme, **synchronized rises and falls** in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time - the "perfect storm" scenario**"

(Business Week, September 1998)

Correlation



1. 19.10.1987
   Black Monday
2. 16.10.1989
   Berlin Wall
3. 19.08.1991
   Kremlin
4. 17.03.2008, 19.09.2008, 10.10.2008, 13.10.2008, 15.10.2008, 29.10.2008
   Crisis

Correlation

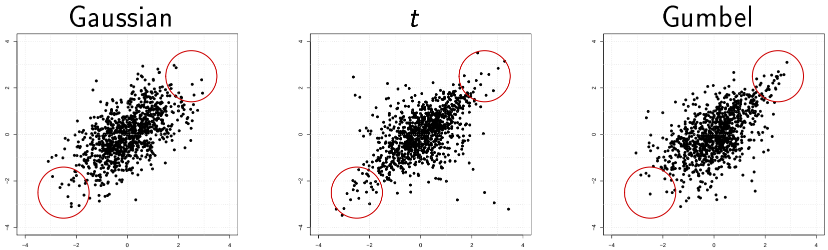


Figure 1: Scatterplots for two distributions with $\rho = 0.4$

- same linear correlation coefficient ($\rho = 0.4$)
- same marginal distributions
- rather big difference

Copula

For a distribution function F with marginals $F_{X_1}, \dots, F_{X_d}$, there exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \dots, x_d) = \mathbf{C}\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\}.$$

A little bit of history

- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions
- 1959: The word **copula** appears for the first time (*Abe Sklar*)
- 1999: Introduced to financial applications (*Paul Embrechts*, *Alexander McNeil*, *Daniel Straumann* in RISK Magazine)
- 2000: Paper by *David Li* in *Journal of Derivatives* on application of copulae to CDO
- 2006: Several insurance companies, banks and other financial institutions apply copulae as a risk management tool

Applications

Practical Use:

1. medicine (Vandenhende (2003))
2. hydrology (Genest and Favre (2006))
3. biometrics (Wang and Wells (2000, JASA), Chen and Fan (2006, CanJoS))
4. economics
    - portfolio selection (Patton (2004, JoFE), Xu (2004, PhD thesis), Hennessy and Lapan (2002, MathFin))
    - time series (Chen and Fan (2006a, 2006b, JoE), Fermanian and Scaillet (2003, JoR), Lee and Long (2005, JoE))
    - risk management (Junker and May (2002, EJ), Breyman et. al. (2003, QF))

Special Copulas

Theorem

Let C be a copula. Then for every $(u_1, u_2) \in [0,1]^2$

$$\max(u_1 + u_2 - 1, 0) \leq C(u_1, u_2) \leq \min(u_1, u_2),$$

*where bounds are called **lower and upper Fréchet-Höffdings bounds**. When they are copulas they represent perfect negative and positive dependence respectively.*

The simplest copula is **product copula**

$$\Pi(u_1, u_2) = u_1 u_2$$

characterize the case of independence.

Copula Classes

1. elliptical
   - implied by well-known multivariate df's (Normal, t), derived through Sklar's theorem
   - do not have closed form expressions and are restricted to have radial symmetry
2. Archimedean

$$C(u_1, u_2) = \phi^{-1}\{\phi(u_1) + \phi(u_2)\}$$

   - allow for a great variety of dependence structures
   - closed form expressions
   - several useful methods for multivariate extension
   - not derived from mv df's using Sklar's theorem

Copula Examples 1

Gaussian copula

$$
\begin{aligned}
C_\delta^G(u_1, u_2) &= \mathbf{\Phi}_\delta\{\mathbf{\Phi}^{-1}(u_1), \mathbf{\Phi}^{-1}(u_2)\} \\
&= \int_{-\infty}^{\mathbf{\Phi}^{-1}(u_1)} \int_{-\infty}^{\mathbf{\Phi}^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\delta^2}} \exp\left\{\frac{-(s^2 - 2\delta st + t^2)}{2(1-\delta^2)}\right\} ds\, dt,
\end{aligned}
$$

- Gaussian copula contains the dependence structure
- *normal* marginal distribution + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distribution + Gaussian copula = meta-Gaussian distributions
- allows to generate joint symmetric dependence, but no tail dependence

Copula Examples 2

Gumbel copula

$$C_\theta^{Gu}(u_1, u_2) = \exp\left\{-\left[(-\log u_1)^{1/\theta} + (-\log u_2)^{1/\theta}\right]^\theta\right\}.$$

- for $\theta > 1$ allows to generate dependence in the upper tail
- for $\theta = 1$ reduces to the product copula
- for $\theta \to \infty$ obtain Frèchet-Hoeffding upper bound

$$C_\theta(u_1, u_2) \stackrel{\theta\to\infty}{\to} \min(u_1, u_2)$$

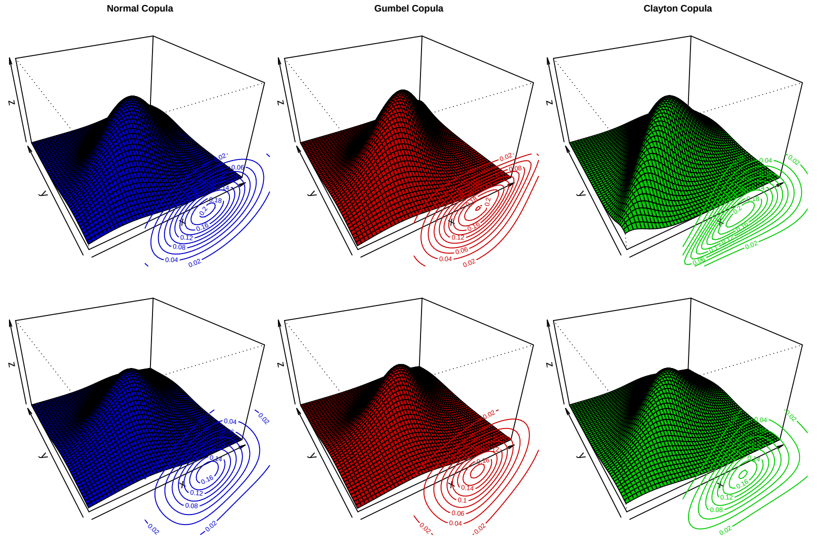
- the only extreme value Archimedean copula

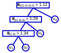
Copula Examples 3

Clayton copula

$$C_\theta^{Cl}(u_1, u_2) = \left[\max\left(u_1^{-\theta} + u_2^{-\theta} - 1, 0\right)\right]^{-\frac{1}{\theta}}$$

- dependence becomes maximal when $\theta \to \infty$
- independence is achieved when $\theta = 0$
- the distribution tends to the lower Frèchet-Hoeffding bound when $\theta \to 1$
- allows to generate asymmetric dependence and lower tail dependence, but no upper tail dependence
- the only Archimedean copula with truncated property


Normal Copula
Gumbel Copula
Clayton Copula



Dependencies, Linear Correlation

$$\delta(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}}.$$

- Sensitive to outliers
- Measures the 'average dependence' between X_1 and X_2
- Invariant under strictly increasing linear transformations
- May be misleading in situations where multivariate df is not elliptical

Dependencies, Kendall's tau

Definition

If F is continuous bivariate cdf and let (X_1, X_2), (X_1', X_2') be independent random pairs with distribution F. Then **Kendall's tau** is

$$\tau = P\{(X_1 - X_1')(X_2 - X_2') > 0\} - P\{(X_1 - X_1')(X_2 - X_2') < 0\}$$

- Less sensitive to outliers
- Measures the 'average dependence' between X and Y
- Invariant under strictly increasing transformations
- Depends only on the copula of (X_1, X_2)
- For elliptical copulae: $\delta(X_1, X_2) = \sin\left(\frac{\pi}{2}\tau\right)$

Dependencies, Spearmans's rho

Definition

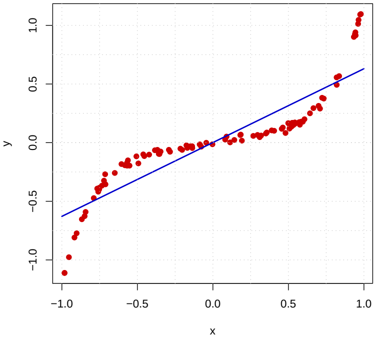
If F is a continuous bivariate cumulative distribution function with marginal F_1 and F_2 and let $(X_1, X_2) \sim F$. Then **Spearmans's rho** is a correlation between $F_1(X_1)$ and $F_2(X_2)$

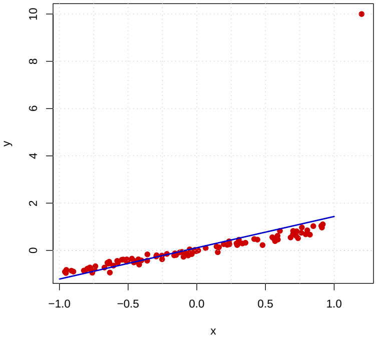
$$\rho = \frac{Cov\{F_1(X_1), F_2(X_2)\}}{\sqrt{Var\{F_1(X_1)\}Var\{F_2(X_2)\}}}.$$

- Less sensitive to outliers
- Measures the 'average dependence' between X_1 and X_2
- Invariant under strictly increasing transformations
- Depends only on the copula of (X_1, X_2)
- For elliptical copulae: $\delta(X_1, X_2) = 2\sin\left(\frac{\pi}{6}\rho\right)$

$$\delta = 0.892, \quad \tau = 0.956, \quad \rho = 0.996$$

$$\delta = 0.659, \quad \tau = 0.888, \quad \rho = 0.982$$





Dependencies, Examples

Gaussian copula

$$\begin{aligned}\rho &= \frac{6}{\pi}\,\text{arcsin}\,\frac{\delta}{2},\\ \tau &= \frac{2}{\pi}\,\text{arcsin}\,\delta,\end{aligned}$$

where δ is a linear correlation coefficient.
Gumbel copula

$$\begin{aligned}\rho &- \text{no closed form},\\ \tau &= 1-\frac{1}{\theta}.\end{aligned}$$

Multivariate Copula Definition

Definition

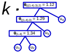
The **copula** is a multivariate distribution with all univariate margins being $U(0,1)$.

Theorem (Sklar, 1959)

Let $X_1, \dots, X_k$ be random variables with marginal distribution functions $F_1, \dots, F_k$ and joint distribution function F. Then there exists a k-dimensional copula $C : [0,1]^k \to [0,1]$ such that $\forall x_1, \dots, x_k \in \overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \dots, x_k) = C\{F_1(x_1), \dots, F_k(x_k)\} \qquad (1)$$

If the margins $F_1, \dots, F_k$ are continuous, then C is unique. Otherwise C is uniquely determined on $F_1(\overline{\mathbb{R}}) \times \cdots \times F_k(\overline{\mathbb{R}})$ Conversely, if C is a copula and $F_1, \dots, F_k$ are distribution functions, then the function F defined in (1) is a joint distribution function with margins $F_1, \dots, F_k$.



Copula Density

Several theorems provides existence of derivatives of copulas, having them copula density is defined as

$$c(u_1, \ldots, u_k) = \frac{\partial^n C(u_1, \ldots, u_k)}{\partial u_1 \ldots \partial u_k}.$$

Joint density function based on copula

$${}_c f(x_1, \ldots, x_k) = c\{F_1(x_1), \ldots, F_k(x_k)\} \cdot f_1(x_1) \ldots f_k(x_k),$$

where $f_1(\cdot), \ldots, f_k(\cdot)$ are marginal density functions.

Special Copulas

Theorem

Let C be a copula. Then for every $(u_1, \ldots, u_k) \in [0,1]^k$

$$\max\left(\sum_{i=1}^{k} u_i + 1 - k, 0\right) \leq C(u_1, \ldots, u_k) \leq \min(u_1, \ldots, u_k),$$

*where bounds are called **lower and upper Fréchet-Höffdings bounds**. When they are copulas they represent perfect negative and positive dependence respectively.*

The simplest copula is **product copula**

$$\Pi(u_1, \ldots, u_k) = \prod_{i=1}^{k} u_i$$

characterize the case of independence.

Simulation

Frees and Valdez, (1998, NAAJ), Whelan, (2004, QF), Marshal and Olkin, (1988, JASA)

Conditional inversion method:

Let $C = C(u_1, \dots, u_k)$, $C_i = C(u_1, \dots, u_i, 1, \dots, 1)$ and $C_k = C(u_1, \dots, u_k)$. Conditional distribution of U_i is given by

$$\begin{aligned} C_i(u_i|u_1, \dots, u_{i-1}) &= P\{U_i \leq u_i | U_1 = u_1 \dots U_{i-1} = u_{i-1}\} \\ &= \frac{\partial^{i-1} C_i(u_1, \dots, u_i)}{\partial u_1 \dots \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C_{i-1}(u_1, \dots, u_{i-1})}{\partial u_1 \dots \partial u_{i-1}} \end{aligned}$$

- Generate i.r.v. $v_1, \dots, v_k \sim U(0,1)$
- Set $u_1 = v_1$
- $u_i = C_k^{-1}(v_i|u_1, \dots, u_{i-1}) \; \forall i = \overline{2, k}$

Main Idea

- combine interpretability with flexibility without loosing statistical precision
- determine the optimal structure of HAC
- convenient and useful probabilistic properties of the HAC

Recall Archimedean Copula

Multivariate Archimedean copula $C : [0,1]^d \to [0,1]$ defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \qquad (2)$$

where $\phi : [0,\infty) \to [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.

Example

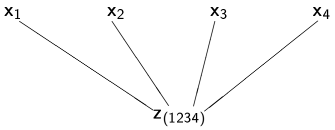
$$\begin{aligned} \phi_{Gumbel}(u,\theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \le \theta < \infty \\ \phi_{Clayton}(u,\theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1,\infty)\backslash\{0\} \end{aligned}$$

Disadvantages: too restrictive: single parameter, exchangeable

Hierarchical Archimedean Copulas

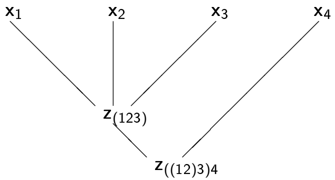
Simple AC with s=(1234)

$$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$$



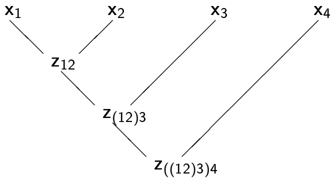
AC with s=((123)4)

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$$



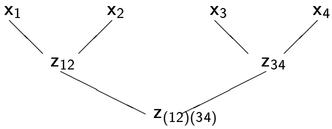
Fully nested AC with s=(((12)3)4)

$$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$$



Partially Nested AC with s=((12)(34))

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$$



Hierarchical Archimedean Copula

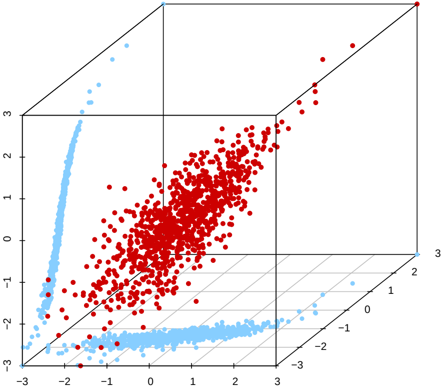


Figure 2: Scatterplot of the $C_{Gumbel}[C_{Gumbel}\{\Phi(x_1), t_2(x_2); \theta_1 = 2\}, \Phi(x_3); \theta_2 = 10]$, $s = ((12)3)$

Hierarchical Archimedean Copula

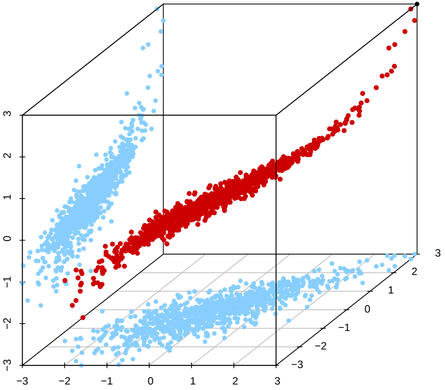


Figure 3: Scatterplot of the $C_{Gumbel}[\Phi(x_2), C_{Gumbel}\{t_2(x_1), \Phi(x_3); \theta_1 = 2\}; \theta_2 = 10]$, $s = (2(13))$

Hierarchical Archimedean Copula

Advantages of HAC:

- flexibility and wide range of dependencies:
  for $d = 10$ more than $2.8 \cdot 10^8$ structures
- dimension reduction:
  $d - 1$ parameters to be estimated
- subcopulas are also HAC

Theoretical motivation

Let M be the cdf of a positive random variable and ϕ denotes its Laplace transform, i.e. $\phi(t) = \int_0^\infty e^{-tw} dM(w)$. For an arbitrary cdf F there exists a unique cdf G, such that

$$F(x) = \int_0^\infty G^\alpha(x) dM(\alpha) = \phi\{-\ln G(x)\}.$$

Now consider a k-variate cumulative distribution function F with margins $F_1, \dots, F_d$. Then it holds for $G_j = \exp\{-\phi^{-1}(F_j)\}$ that

$$\int_0^\infty G_1^\alpha(x_1) \cdot \dots \cdot G_d^\alpha(x_d) dM(\alpha) = \phi\left\{-\sum_{i=1}^{d} \ln G_i(x_i)\right\} = \phi\left[\sum_{i=1}^{d} \phi^{-1}\{F_i(x_i)\}\right].$$

$$C(u_1, \dots, u_d) =$$

$$\int_0^\infty \dots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1, \alpha_2)\ G_3^{\alpha_2}(u_3) dM_2(\alpha_2, \alpha_3) \dots G_d^{\alpha_{d-1}}(u_d) dM_{d-1}(\alpha_{d-1})$$

Recovering the structure (theory)

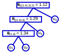
To guarantee that C is a HAC we assume that $\phi^{-1}_{d-i} \circ \phi_{d-j} \in \mathcal{L}^*$, $i < j$ with

$$\mathcal{L}^* = \{\omega : [0,\infty) \to [0,\infty) \,|\, \omega(0) = 0,\ \omega(\infty) = \infty,\ (-1)^{j-1}\omega^{(j)} \geq 0,\ j \geq 1\}.$$

☞ for most of the generator functions the parameters should decrease from the lowest level to the highest

Theorem

Let F be an arbitrary multivariate distribution function based on HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.



$$C(u_1, \dots, u_6) = C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}].$$

The bivariate marginal distributions are then given by

$$\begin{array}{lll}
(U_1, U_2) \sim C_2(\cdot,\cdot), & (U_2, U_3) \sim C_1(\cdot,\cdot), & (U_3, U_5) \sim C_3(\cdot,\cdot), \\
(U_1, U_3) \sim C_1(\cdot,\cdot), & (U_2, U_4) \sim C_1(\cdot,\cdot), & (U_3, U_6) \sim C_3(\cdot,\cdot), \\
(U_1, U_4) \sim C_1(\cdot,\cdot), & (U_2, U_5) \sim C_1(\cdot,\cdot), & (U_4, U_5) \sim C_4(\cdot,\cdot), \\
(U_1, U_5) \sim C_1(\cdot,\cdot), & (U_2, U_6) \sim C_1(\cdot,\cdot), & (U_4, U_6) \sim C_3(\cdot,\cdot), \\
(U_1, U_6) \sim C_1(\cdot,\cdot), & (U_3, U_4) \sim C_3(\cdot,\cdot), & (U_5, U_6) \sim C_3(\cdot,\cdot).
\end{array}$$

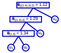
$$\mathcal{C}_2\{\mathbb{N}(C)\} = \{C_1(\cdot,\cdot), C_2(\cdot,\cdot), C_3(\cdot,\cdot), C_4(\cdot,\cdot)\}.$$

- each variable belongs to at least one bivariate margin C_1 $\rightsquigarrow$ the distribution of $u_1, \ldots, u_6$ has C_1 at the top level.
- C_3 covers the largest set of variables u_3, u_4, u_5, u_6 $\rightsquigarrow$ C_3 is at the top level of the subcopula containing u_3, u_4, u_5, u_6.

$$U_1, \ldots, U_6 \sim C_1\{u_1, u_2, C_3(u_3, u_4, u_5, u_6)\}.$$

- C_2 and C_4 and they join u_1, u_2 and u_4, u_5 respectively.

$$(U_1, \ldots, U_6) \sim C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}]$$



Let for each bivariate copula $C^* \in \mathcal{C}_2\{\mathbb{N}(C)\}$, $I(C)$ be the set of indices $i \in \{1, \dots, k\}$ such that $(U_i, U_j) \sim C^*$ for at least one $j \in \{1, \dots, k\} \backslash \{i\}$.

$I(C_1) = \{1, \dots, 6\}$, $I(C_2) = \{1, 2\}$, $I(C_3) = \{3, 4, 5, 6\}$, $I(C_4) = \{4, 5\}$

The family of sets $I(C^*)$, as C^* ranges over $\mathcal{C}_2\{\mathbb{N}(C)\}$, is partially ordered by inclusion

$$I(C_1) \supset \begin{cases} I(C_2), \\ I(C_3) \supset I(C_4). \end{cases}$$

Recovering the structure (practice)

$$\begin{array}{l} (12) \rightsquigarrow \widehat{\theta}_{12} \\ (13) \rightsquigarrow \widehat{\theta}_{13} \\ (14) \rightsquigarrow \widehat{\theta}_{14} \\ (23) \rightsquigarrow \widehat{\theta}_{23} \\ (24) \rightsquigarrow \widehat{\theta}_{24} \\ (34) \rightsquigarrow \widehat{\theta}_{34} \\ \hline (123) \rightsquigarrow \widehat{\theta}_{123} \\ (124) \rightsquigarrow \widehat{\theta}_{124} \\ (234) \rightsquigarrow \widehat{\theta}_{234} \\ (134) \rightsquigarrow \widehat{\theta}_{134} \\ (1234) \rightsquigarrow \widehat{\theta}_{1234} \end{array}$$

best fit (13) $\rightsquigarrow$

$$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$$

$$\begin{array}{c} (13)2 \rightsquigarrow \widehat{\theta}_{(13)2} \\ (13)4 \rightsquigarrow \widehat{\theta}_{(13)4} \\ 24 \rightsquigarrow \widehat{\theta}_{24} \\ \hline (13)24 \rightsquigarrow \widehat{\theta}_{(13)24} \end{array}$$

best fit ((13)4) $\rightsquigarrow$

$$z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}$$

$$((13)4)2 \rightsquigarrow \widehat{\theta}_{((13)4)2}$$

Estimation: multistage MLE with nonparametric and parametric margins

Criteria for grouping: goodness-of-fit tests, parameter-based method, etc.

Estimation Issues - Margins

$$
\begin{aligned}
F_j(x;\widehat{\alpha}_j) &= F_j\left\{x; \mathit{arg}\max_{\alpha}\sum_{i=1}^{n}\log\ f_j(X_{ji},\alpha)\right\},\\
\widehat{F}_j(x) &= \frac{1}{n+1}\sum_{i=1}^{n}\mathbf{I}(X_{ji}\leq x),\\
\widetilde{F}_j(x) &= \frac{1}{n+1}\sum_{i=1}^{n}K\left(\frac{x-X_{ji}}{h}\right)
\end{aligned}
$$

for $j = 1, \ldots, k$, where $\varkappa : \mathbb{R} \to \mathbb{R}$, $\int \varkappa = 1$, $K(x) = \int_{-\infty}^{x} \varkappa(t)dt$ and $h > 0$ is the bandwidth.

$$\check{F}_j(x) \in \{\widehat{F}_j(x), \widetilde{F}_j(x), F_j(x;\widehat{\alpha}_j)\}$$

Estimation Issues - Multistage Estimation

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\theta}_1^\top}, \dots, \frac{\partial \mathcal{L}_p}{\partial \boldsymbol{\theta}_p^\top}\right)^\top = \mathbf{0},$$

$$\begin{aligned} \text{where} \quad \mathcal{L}_j &= \sum_{i=1}^{n} l_j(\mathbf{X}_i) \\ l_j(\mathbf{X}_i) &= \log\left(c(\{\phi_\ell, \boldsymbol{\theta}_\ell\}_{\ell=1,\dots,j};\, s_j)\left[\{\check{F}_m(x_{mi})\}_{m\in s_j}\right]\right) \\ &\text{for } j = 1, \dots, p. \end{aligned}$$

Theorem

Under regularity conditions, estimator $\widehat{\boldsymbol{\theta}}$ is consistent and

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{a}{\sim} N(\mathbf{0}, \mathbf{B}^{-1}\Sigma\mathbf{B}^{-1})$$

Criteria for grouping

Alternatives:

- goodness-of-fit tests $\rightsquigarrow$ to be discussed
  - dimension dependent
  - KS type tests are difficult to implement
  - possible choice $\rightsquigarrow$ Chen et al. (2004, WP of LSE), Fermanian (2005, JMA)
- distance measures
  - dimension dependent
- parameter-based methods

  Note that, if the true structure is (123) then

  $$\theta_{(12)} = \theta_{(13)} = \theta_{(23)} = \theta_{(123)}.$$

  - heurithtic methods
  - test-based methods
- tests on exchangeability

Criteria for grouping using GOF

$$H_0: \quad C = C_0, \quad \text{against} \quad H_1: \quad C \neq C_0.$$

probability integral transform, Rosenblatt (1952, AMS)

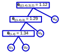
$$Y_{1i} = \check{F}_1(x_{1i}), \quad Y_{ji} = C(\phi, \widehat{\theta}, s)\{\check{F}_j(x_{ji}) | \check{F}_1(x_{1i}), \ldots, \check{F}_{j-1}(x_{j-1,i})\}$$

$$\widehat{W}_i = \sum_{j=1}^{d} \{\Phi^{-1}(Y_{ji})\}^2, \quad \widehat{g}_W(w) = \frac{1}{nh} \sum_{i=1}^{n} K_h\{w, F_{\chi^2_d}(\widehat{W}_i)\},$$

$$\widehat{J}_n = \int_0^1 \{\widehat{g}_W(w) - 1\}^2 dw$$

test statistic (Chen et al. 2004)

$$T_n = \frac{(n\sqrt{h}\widehat{J}_n - c_n)}{\sigma} \to N(0,1).$$



Criteria for grouping based on θ's

I. For all subsets perform tests of the kind

$$
\begin{aligned}
H_0 &: \quad \theta_{(12)} = \theta_{(13)} = \theta_{(23)} = \theta_{(123)} \\
H_1 &: \quad \text{at least one equality is not fulfilled}
\end{aligned}
$$

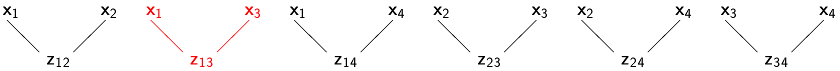
II.

$$\Delta = \min_{I_{ki},\, |I_{ki}| \geq 3} \; \max_{I_{|I_{ki}|,j} \subset I_{ki}} |\theta(I_{ki}) - \theta(I_{|I_{ki}|,j})|,$$

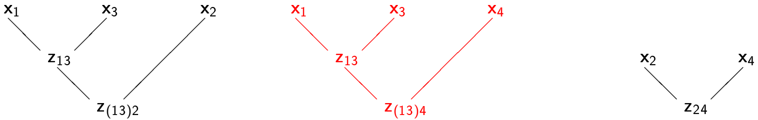
where $j = 1, \ldots, 2^{|I_{ki}|} - |I_{ki}| - 1$ and $\{I_{ki}\}_{i=1,\ldots 2^k - k - 1}$ denote the subsets of the initial set of size k, excluding empty set and single element sets.

$$I^* \;=\; \begin{cases} I_\Delta, & \Delta \leq \delta \\ \max_{I_{ki},\, |I_{ki}|=2} \; \theta(I_{ki}), & \Delta > \delta \end{cases}.$$

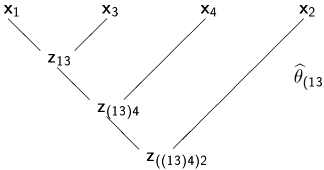
Recovering the structure (easy practice)



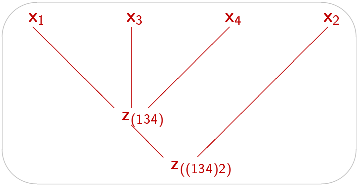
$$\max\{\widehat{\theta}_{12}, \widehat{\theta}_{13}, \widehat{\theta}_{14}, \widehat{\theta}_{23}, \widehat{\theta}_{24}, \widehat{\theta}_{34}\} = \widehat{\theta}_{13} \quad \Rightarrow$$



$$\max\{\widehat{\theta}_{(13)2}, \widehat{\theta}_{(13)4}, \widehat{\theta}_{24}\} = \widehat{\theta}_{(13)4} \quad \Rightarrow$$



$$\widehat{\theta}_{(13)} \approx \widehat{\theta}_{((13)4)} \Rightarrow$$



Simulation, I

Method	Copula structure(s)	%	KL	Kendall τ	λ_U	λ_L	time (in sec)
Gauss			0.282 (0.040)	0.078 (0.029)	3.026 (0.000)	0.000 (0.000)	17.682 (9.924)
t			0.197 (0.029)	0.069 (0.027)	1.346 (0.160)	1.773 (0.141)	40.030 (24.234)
sAC	$(12345)_{2.32}$	100.0	0.803 (0.071)	0.525 (0.010)	0.471 (0.008)	0.000 (0.000)	1.251 (1.035)
$\tau_{\triangle\tau>0}$	$((45)_{3.01}(123)_{4.13})_{2.23}$	99.5	0.017 (0.012)	0.181 (0.072)	0.172 (0.069)	0.000 (0.000)	0.591 (0.315)
	$((45)_{3.04}(1(23)_{4.45})_{4.35})_{2.21}$	0.3					
	$((45)_{2.97}(2(13)_{4.2})_{4.09})_{2.09}$	0.2					
τ_b	$((45)_{3}(1(23)_{4.11})_{4.1})_{2.23}$	35.5	0.017 (0.012)	0.181 (0.072)	0.172 (0.069)	0.000 (0.000)	0.590 (0.349)
	$((45)_{3}(3(12)_{4.13})_{4.13})_{2.23}$	32.5					
	$((45)_{3.02}(2(13)_{4.13})_{4.13})_{2.23}$	32.0					
Chen	$(23(145)_{2.18})_{2.02}$	16.6	0.739 (0.276)	0.555 (0.146)	0.501 (0.129)	0.000 (0.000)	419.209 (122.406)
	$(45(123)_{4})_{2.01}$	16.5					
	$(13(245)_{2.18})_{2.01}$	15.0					
θ	$(2(1345)_{2.24})_{1.78}$	29.5	0.813 (0.348)	0.650 (0.190)	0.598 (0.172)	0.000 (0.000)	7.433 (3.859)
	$(3(1245)_{2.24})_{1.78}$	28.7					
	$(1(2345)_{2.24})_{1.78}$	26.0					
θ_b	$((45)_{3.01}(1(23)_{4.12})_{3.91})_{2.28}$	35.7	0.021 (0.007)	0.211 (0.056)	0.202 (0.053)	0.000 (0.000)	0.880 (0.493)
	$((45)_{3}(3(12)_{4.11})_{3.91})_{2.28}$	33.0					
	$((45)_{3.01}(2(13)_{4.11})_{3.9})_{2.28}$	31.3					
θ_{ba}	$((45)_{3.01}(1(23)_{4.13})_{3.89})_{2.27}$	32.2	0.021 (0.007)	0.211 (0.056)	0.202 (0.053)	0.000 (0.000)	0.847 (0.461)
	$((45)_{3}(3(12)_{4.12})_{3.89})_{2.28}$	28.0					
	$((45)_{3.01}(2(13)_{4.12})_{3.89})_{2.28}$	27.4					
θ_{PML}	$((1(23)_{4.14})_{3.98}(45)_{3})_{1.99}$	26.3	-0.003 (0.002)	0.051 (0.028)	0.048 (0.028)	0.000 (0.000)	0.537 (0.082)
	$((2(13)_{4.14})_{3.99}(45)_{3})_{1.99}$	25.6					
	$((3(12)_{4.14})_{3.98}(45)_{3})_{1.98}$	23.5					

Table 1: Model fit for the true structure $((123)_4(45)_3)_2$.

Simulation, I

	Structure $((123)_4(45)_3)_2$			
	θ_3	θ_2	θ_1	Time (in s)
true	4.00000	3.00000	2.00000	
MultiStage (mean)	4.01483	3.01093	2.27715	0.24230
MultiStage (sd)	0.11083	0.11285	0.08478	0.01265
MultiStageRec (mean)	4.02814	3.01090	1.96754	0.49618
MultiStageRec (sd)	0.10342	0.11283	0.05717	0.03259
Full (mean)	4.00234	3.01029	2.00294	0.94994
Full (sd)	0.10028	0.11159	0.05841	0.06004

Table 2: The average parameters, the empirical standard deviation and computational times for multistage ML, multistage ML with reestimation and full ML estimation based on 1000 simulated samples of size 1000.

Simulation, II

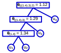
Method	Copula structure(s)	%	KL	Kendall τ	λ_U	λ_L	time (in sec)
Gauss			0.288 (0.039)	0.641 (0.029)	3.088 (0.000)	0.000 (0.000)	15.059 (8.442)
t			0.197 (0.027)	0.644 (0.028)	1.455 (0.144)	1.850 (0.147)	37.261 (22.079)
sAC	$(12345)_{2.37}$	100.0	0.494 (0.067)	0.434 (0.015)	0.394 (0.013)	0.000 (0.000)	1.113 (0.865)
$\tau_{\triangle\tau>0}$	$(5(12(34)_{4.02})_{3.34})_{2.17}$	96.2	0.032 (0.020)	0.165 (0.057)	0.150 (0.052)	0.000 (0.000)	0.536 (0.419)
	$(5(1(2(34)_{3.99})_{3.52})_{3.42})_{2.22}$	2.1					
	$(5(2(1(34)_{4.03})_{3.57})_{3.44})_{2.18}$	1.7					
τ_b	$(5(2(1(34)_{4.02})_{3.35})_{3.34})_{2.17}$	51.2	0.032 (0.020)	0.164 (0.057)	0.150 (0.052)	0.000 (0.000)	0.523 (0.361)
	$(5(1(2(34)_{4.01})_{3.36})_{3.36})_{2.18}$	47.9					
	$(5((34)_4(12)_{3.28})_{3.28})_{2.16}$	0.9					
Chen	$(24(135)_{2.19})_{2.1}$	11.2	0.450 (0.135)	0.494 (0.087)	0.450 (0.080)	0.000 (0.000)	382.225 (116.211)
	$(25(134)_{3.2})_{2.1}$	10.7					
	$(15(234)_{3.2})_{2.11}$	10.7					
θ	$(5(4(123)_3)_{2.63})_{1.73}$	44.9	0.145 (0.031)	0.299 (0.054)	0.284 (0.054)	0.000 (0.000)	6.071 (3.026)
	$(5(3(124)_3)_{2.63})_{1.73}$	44.7					
	$(5(1234)_{3.08})_{1.76}$	10.2					
θ_b	$(5((34)_{3.99}(12)_{3.08})_{3.08})_{1.78}$	38.0	0.031 (0.016)	0.221 (0.058)	0.216 (0.058)	0.000 (0.000)	0.706 (0.382)
	$(5(2(1(34)_{4.02})_{3.08})_{2.6})_{1.75}$	31.8					
	$(5(1(2(34)_{4.01})_{3.07})_{2.59})_{1.75}$	30.2					
θ_{ba}	$(5(12(34)_{3.99})_{3.08})_{1.78}$	38.0	0.031 (0.016)	0.221 (0.058)	0.216 (0.058)	0.000 (0.000)	0.665 (0.340)
	$(5(2(1(34)_{4.02})_{3.08})_{2.6})_{1.75}$	31.8					
	$(5(1(2(34)_{4.01})_{3.07})_{2.59})_{1.75}$	30.2					
θ_{PML}	$(5((12)_{3.09}(34)_{3.98})_{2.89})_{1.99}$	32.8	-0.002 (0.003)	0.054 (0.023)	0.049 (0.022)	0.000 (0.000)	3.288 (2.100)
	$(5(2(1(34)_{4.04})_{3.1})_{2.99})_2$	21.0					
	$(5(1(2(34)_{4.02})_{3.09})_{2.98})_2$	19.7					

Table 3: Model fit for the true structure $((12(34)_4)_35)_2$.

Simulation, II

	Structure $((12(34)_4)_3 5)_2$			
	θ_3	θ_2	θ_1	Time (in s)
true	4.00000	3.00000	2.00000	
MultiStage (mean)	3.98305	3.03973	1.77748	0.25594
MultiStage (sd)	0.14800	0.08496	0.05846	0.02065
MultiStageRec (mean)	3.98301	2.99587	2.00394	1.99505
MultiStageRec (sd)	0.14801	0.07880	0.06110	0.37285
Full (mean)	3.98041	3.00407	2.00520	2.74066
Full (sd)	0.14158	0.07045	0.06109	0.32671

Table 4: The average parameters, the empirical standard deviation and computational times for multistage ML, multistage ML with reestimation and full ML estimation based on 1000 simulated samples of size 1000.



Missspesification

Let $H(x_1, \dots, x_k)$ – true df with density h. Since H is unknown we specify $F(x_1, \dots, x_k, \boldsymbol{\eta})$ with density f.

- F is correctly specified:
  $\exists \boldsymbol{\eta}_0 : F(x_1, \dots, x_k, \boldsymbol{\eta}_0) = H(x_1, \dots, x_k),\ \forall (x_1, \dots, x_k)$ then $\widehat{\boldsymbol{\eta}}$ is consistent for $\boldsymbol{\eta}_0$.
- F is not correctly specified:
  $\nexists \boldsymbol{\eta}_0 : F(x_1, \dots, x_k, \boldsymbol{\eta}_0) = H(x_1, \dots, x_k),\ \forall (x_1, \dots, x_k)$, then $\widehat{\boldsymbol{\eta}}$ is an estimator for $\boldsymbol{\eta}_*$ which minimizes the Kullback–Leibler divergence between f and h as

$$\mathcal{K}(h, f, \boldsymbol{\eta}) = \boldsymbol{E}_h\{\log[h(x_1, \dots, x_k)/f(x_1, \dots, x_k, \boldsymbol{\eta})]\},$$

Missspesification, I

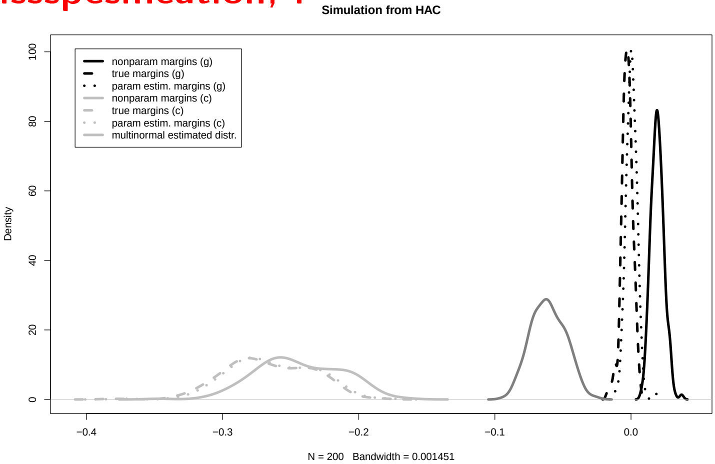


Figure 4: Kullback-Leibler divergences for the simulated samples, HAC, $\theta_1 = 2.0, \theta_2 = 1.5, N = 200, n = 1000$

Missspesification, II

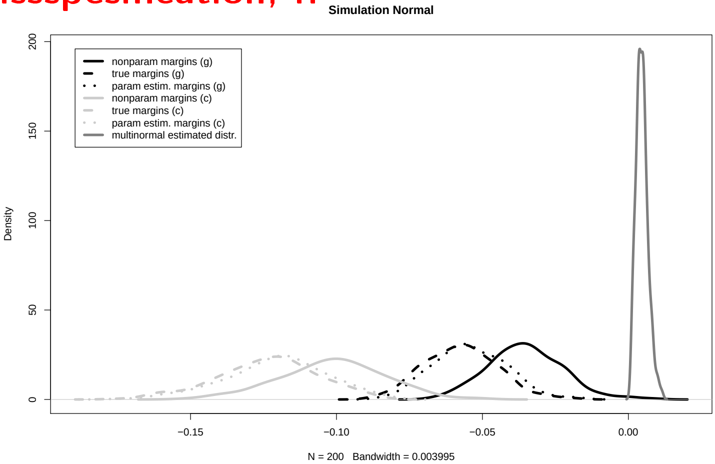


Figure 5: Kullback-Leibler divergences for the simulated samples, HAC, $\Sigma = (0.1, 0.3, 0.5)^\top$, $N = 200$, $n = 1000$

Data and Copula

- daily returns of Apple (APL), Hewlett Packard (HP) and Microsoft (MSFT)
- timespan = [04.01.2006 - 04.11.2009] ($n = 1000$)
- Gumbel and Clayton generators
- AR(1)–GARCH(1,1)-residuals are conditionally distributed with estimated copula

$$\varepsilon \sim C\{F_1(x_1), \ldots, F_d(x_d); \theta_t\}$$

where $F_1, \ldots, F_d$ are marginal distributions taking to be nonparametrically and θ_t are the copula parameters.

Data and Copula

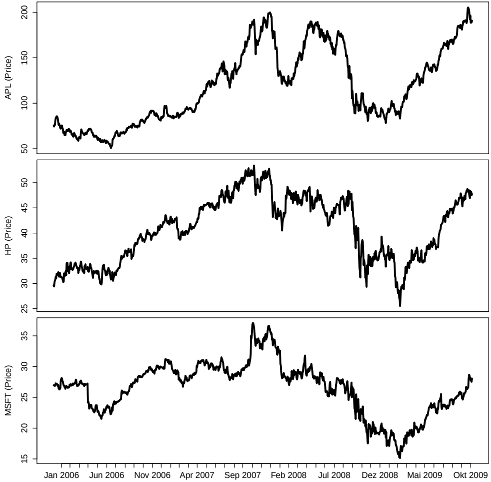
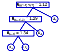


Figure 6: Prices



Data and Copula

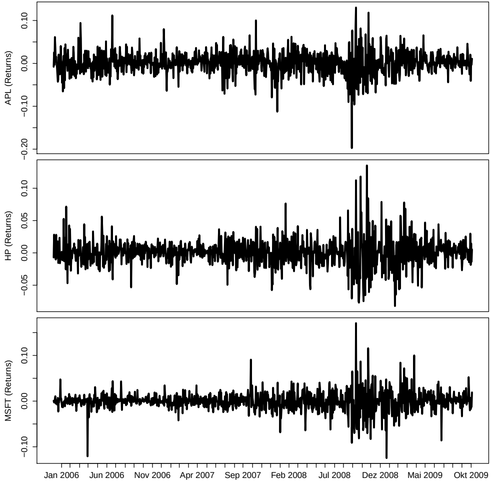


Figure 7: Returns

AR(1)–GARCH(1,1)

	μ	ω_1	γ_0	γ_1	δ_1	BL	KS
APL	2.481$e{-}3$	3.941$e{-}4$	1.659$e{-}5$	0.0764	9.008$e{-}1$	0.516	3.867$e{-}3$
	7.337$e{-}4$	3.362$e{-}2$	7.338$e{-}6$	0.0165	2.292$e{-}2$		
HP	1.304$e{-}4$	-8.267$e{-}2$	4.125$e{-}6$	0.0657	9.241$e{-}1$	0.650	2.274$e{-}3$
	4.965$e{-}4$	3.315$e{-}2$	1.694$e{-}6$	0.0124	1.359$e{-}2$		
MSFT	3.962$e{-}4$	-7.555$e{-}2$	1.349$e{-}5$	0.0783	8.894$e{-}1$	0.263	2.970$e{-}5$
	5.262$e{-}4$	3.563$e{-}2$	4.249$e{-}6$	0.0206	2.800$e{-}2$		

Table 5: Estimation results of fitting univariate AR(1)–GARCH(1,1) processes to the data with the volatility equation $\sigma_t^2 = \gamma_0 + \gamma_1(r_{t-1} - \mu - \omega_1 r_{t-2})^2 + \delta_1 \sigma_{t-1}^2$. Second lines contain the standard deviations of the parameters.

Data and Copula

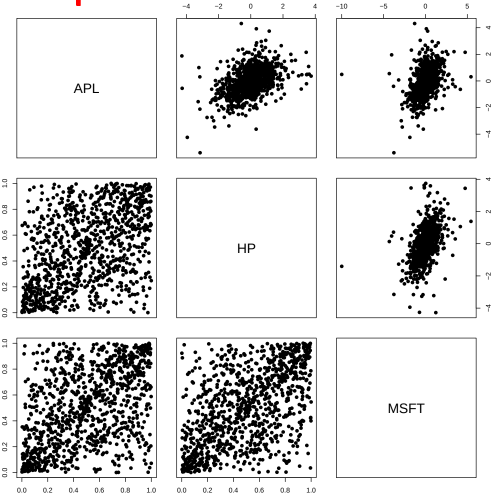


Figure 8: Residuals

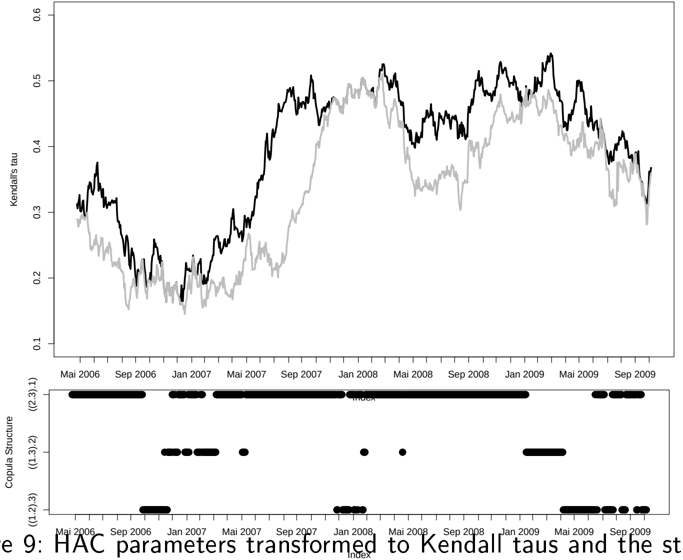


Figure 9: HAC parameters transformed to Kendall taus and the structure from moving window estimation with window length of 100 observations.

VaR

The P&L function is $L_{t+1} = \sum_{i=1}^{3} w_i P_{it}(e^{R_{i,t+1}} - 1)$,
The VaR of at level α is $VaR(\alpha) = F_L^{-1}(\alpha)$

$$\widehat{\alpha}_{\mathbf{w}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{I}\{L_t < \widehat{VaR}_t(\alpha)\}.$$

The distance between $\widehat{\alpha}$ and α

$$e_{\mathbf{w}} = (\widehat{\alpha}_{\mathbf{w}} - \alpha)/\alpha.$$

The performance of models is measured through

$$A_W = \frac{1}{|W|} \sum_{\mathbf{w} \in W} e_{\mathbf{w}}, \qquad D_W = \left\{ \frac{1}{|W|} \sum_{\mathbf{w} \in W} (e_{\mathbf{w}} - A_W)^2 \right\}^{1/2}.$$

VaR

	Gauss			AC			HAC		
α	0.100	0.050	0.010	0.100	0.050	0.010	0.100	0.050	0.010
$\widehat{\alpha}_{w^*}$	0.104	0.052	0.011	0.107	0.058	0.014	0.102	0.057	0.012
$\widehat{\alpha}_{w_1}$	0.094	0.056	0.011	0.096	0.056	0.013	0.095	0.054	0.012
$\widehat{\alpha}_{w_2}$	0.103	0.054	0.011	0.103	0.057	0.013	0.099	0.053	0.012
$\widehat{\alpha}_{w_3}$	0.104	0.053	0.011	0.103	0.055	0.012	0.101	0.052	0.012
$\widehat{\alpha}_{w_4}$	0.103	0.054	0.011	0.105	0.057	0.013	0.099	0.053	0.012
$\widehat{\alpha}_{w_5}$	0.104	0.055	0.011	0.108	0.062	0.014	0.103	0.054	0.012
A_W	0.011	0.019	0.120	0.027	0.092	0.285	-0.005	0.031	0.208
D_W	0.054	0.057	0.045	0.064	0.068	0.088	0.042	0.043	0.069

Table 6: Exceedance ratios for portfolios w^*, $w_i, i = 1, \ldots, 5$, the average exceedance A_W over all portfolios and its standard deviation.

Distribution of HAC

Let $V = C\{F_1(X_1), \dots, F_d(X_d)\}$ and let $K(t)$ denote the distribution function (K-distribution) of the random variable V.

We consider a HAC of the form $C_1\{u_1, C_2(u_2, \dots, u_d)\}$.

Theorem

Let $U_1 \sim U[0,1]$, $V_2 \sim K_2$ and let $P(U_1 \le x, V_2 \le y) = C_1\{x, K_2(y)\}$ with $C_1(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0,1]$. Under certain regularity conditions the distribution function K_1 of the random variable $V_1 = C_1(U_1, V_2)$ is given by

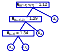
$$
\begin{aligned}
K_1(t) &= t - \int_0^{\phi^{-1}(t)} \phi'\left\{\phi^{-1}(t) + \phi^{-1} \circ K_2 \circ \phi(u) - u\right\} du \\
&\text{for} \quad t \in [0,1].
\end{aligned}
$$

Gumbel copula

$$
\begin{aligned}
\phi_\theta(t) &= \exp(-t^{1/\theta}),\\
\phi_\theta^{-1}(t) &= \{-\log(t)\}^\theta,\\
\phi'_\theta(t) &= -\frac{1}{\theta}\exp(-t^{1/\theta})t^{-1+1/\theta}.
\end{aligned}
$$

Following Genest and Rivest (1993), K for the simple 2-dim Archimedean copula with generator ϕ is given by $K(t) = t - \phi^{-1}(t)\phi'\{\phi^{-1}(t)\}$. Thus

$$
K_2(t,\theta) = t - \frac{t}{\theta}\log(t)
$$



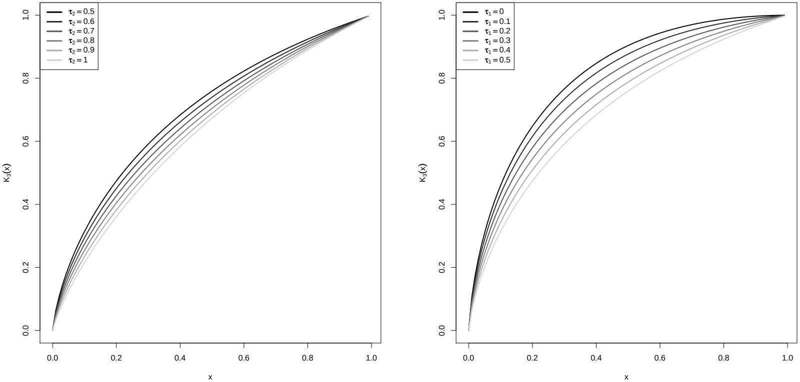


Figure 10: K distribution for three-dimensional HAC with Gumbel generators

Next consider $V_3 = C_3(V_4, V_5)$ with $V_4 = C_4(U_1, \ldots, U_\ell)$ and $V_5 = C_5(U_{\ell+1}, \ldots, U_d)$.

Theorem

Let $V_4 \sim K_4$ and $V_5 \sim K_5$ and $P(V_4 \leq x, V_5 \leq y) = C_3\{K_4(x), K_5(y)\}$ with $C_3(a, b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0, 1]$. Under certain regularity conditions the distribution function K_3 of the random variable $V_3 = C_3(V_4, V_5)$ is given by

$$
\begin{aligned}
K_3(t) &= K_4(t) - \\
&- \int_0^{\phi^{-1}(t)} \phi'\left[\phi^{-1} \circ K_5 \circ \phi(u)\right. \\
&\left. + \phi^{-1} \circ K_4 \circ \phi\{\phi^{-1}(t) - u\}\right] \, d\phi^{-1} \circ K_4 \circ \phi(u)
\end{aligned}
$$

for $t \in [0, 1]$.

Estimation Issues

Nonparametric Estimation

$$\widehat{C}(u_1, \dots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathrm{I}\{\check{F}_j(X_{ji}) \leq u_j\}$$

$$\widetilde{C}(u_1, \dots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} K_j \left\{ \frac{u_j - \check{F}_j(X_{ji})}{h_j} \right\}$$

where $\check{F}_j(x) = \{\widehat{F}_j(x),\ \widetilde{F}_j(x),\ F_j(x, \widehat{\alpha}),\ F_j(x)\}$

Goodness-of-Fit Tests

$$H_0: \quad C \in \mathcal{C}_0, \quad \text{against} \quad H_1: \quad C \notin \mathcal{C}_0,$$

where $\mathcal{C}_0 = \{C_\theta : \theta \in \Theta\}$ is a known parametric family of copulae.

$$\begin{aligned}
S &= n \int_{[0,1]^d} \{\widehat{C}(u_1, \dots, u_d) - C(u_1, \dots, u_d, \widehat{\theta})\}^2 \, d\widehat{C}(u_1, \dots, u_d), \\
T &= \sup_{u_1, \dots, u_d \in [0,1]} \sqrt{n} |\widehat{C}(u_1, \dots, u_d) - C(u_1, \dots, u_d, \widehat{\theta})|, \\
S_K &= n \int_0^1 \{\widehat{K}(v) - K(v, \theta)\}^2 \, dv, \\
T_K &= \sup_{v \in [0,1]} |\widehat{K}(v) - K(v, \theta)|.
\end{aligned}$$

where $\widehat{K}(v) = \frac{1}{n} \sum_{i=1}^n \mathbf{I}\{V_i \leq v\}$.

Simulation Study

1. F : two methods of estimation of margins (parametric and nonparametric);
2. C_0 : hypothesised copula models under H_0 (three models);
3. C : copula model from which the data were generated (three models with 3, 3 and 15 levels of dependence respectively);
4. n : size of each sample drawn from C (two possibilities, $n = 50$ and $n = 150$).

$\rightsquigarrow 2 \times 3 \times (3 + 3 + 15) \times 2 =$ 252 models with 100 repetitions

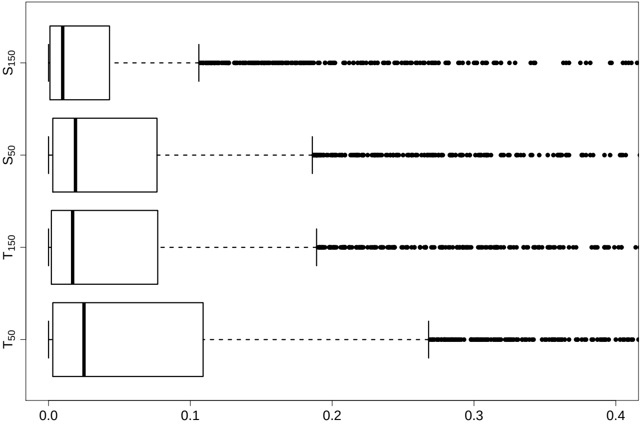


Figure 11: Levels of goodness-of-fit tests for different sample size, for parametric margins.

AC									
		$n = 50$				$n = 150$			
θ		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\theta(0.25)$	HAC	0.88	0.51	0.83	0.38	0.93	0.36	0.90	0.35
	AC	0.88	0.51	0.89	0.50	0.95	0.32	0.90	0.34
	Gauss	0.71	0.29	0.56	0.22	0.69	0.11	0.43	0.08
$\theta(0.5)$	HAC	0.90	0.38	0.94	0.30	0.87	0.35	0.88	0.27
	AC	0.96	0.55	0.95	0.45	0.90	0.45	0.92	0.35
	Gauss	0.76	0.30	0.65	0.19	0.47	0.13	0.31	0.02
$\theta(0.75)$	HAC	0.93	0.29	0.93	0.15	0.89	0.27	0.89	0.10
	AC	0.93	0.29	0.93	0.22	0.90	0.25	0.91	0.13
	Gauss	0.77	0.19	0.65	0.10	0.57	0.11	0.24	0.05

Table 7: Non-rejection rate of the different models, where the sample is drawn from the simple AC

HAC									
		$n = 50$				$n = 150$			
$\boldsymbol{\theta}$		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\theta(0.25, 0.5)$	HAC	0.88	0.29	0.90	0.24	0.96	0.31	0.92	0.26
	AC	0.91	0.26	0.93	0.36	0.54	0.13	0.53	0.07
	Gauss	0.82	0.20	0.69	0.19	0.57	0.14	0.37	0.04
$\theta(0.25, 0.75)$	HAC	0.93	0.21	0.92	0.13	0.88	0.18	0.88	0.09
	AC	0.46	0.14	0.54	0.07	0.00	0.00	0.00	0.00
	Gauss	0.84	0.19	0.71	0.13	0.52	0.10	0.42	0.01
$\theta(0.5, 0.75)$	HAC	0.86	0.31	0.87	0.18	0.91	0.20	0.94	0.08
	AC	0.89	0.36	0.92	0.28	0.44	0.04	0.47	0.02
	Gauss	0.70	0.19	0.55	0.12	0.50	0.11	0.30	0.05

Table 8: Non-rejection rate of the different models, where the sample is drawn from the HAC

Gauss									
		$n = 50$				$n = 150$			
Σ		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\Sigma(0.25, 0.25, 0.75)$	HAC	0.89	0.20	0.93	0.11	0.78	0.08	0.81	0.02
	AC	0.43	0.13	0.47	0.09	0.00	0.00	0.00	0.00
	Gauss	0.88	0.22	0.89	0.12	0.87	0.11	0.86	0.03
$\Sigma(0.25, 0.75, 0.25)$	HAC	0.92	0.20	0.91	0.14	0.76	0.07	0.69	0.04
	AC	0.39	0.12	0.39	0.04	0.00	0.00	0.00	0.00
	Gauss	0.90	0.18	0.87	0.13	0.92	0.12	0.94	0.10
$\Sigma(0.75, 0.25, 0.25)$	HAC	0.89	0.30	0.93	0.16	0.78	0.10	0.75	0.04
	AC	0.51	0.16	0.46	0.07	0.00	0.00	0.00	0.00
	Gauss	0.91	0.28	0.90	0.17	0.88	0.13	0.86	0.06

Table 9: Non-rejection rate of the different models, where the sample is drawn from the Gaussian copula

Data and Copula

- daily returns of Bank of America, Citigroup, Santander
- timespan = [29.09.2000 - 16.02.2001] ($n = 100$)
- ARMA(1,1)-GARCH(1,1)-residuals are conditionally distributed with estimated copula

$$
\begin{aligned}
R_{tj} &= \mu_j + \gamma_j R_{t-1,j} + \zeta_j \sigma_{t-1,j}\varepsilon_{t-1,j} + \sigma_{tj}\varepsilon_{tj}, \\
\sigma^2_{tj} &= \omega_j + \alpha_j \sigma^2_{t-1,j} + \beta_j \sigma^2_{t-1,j}\varepsilon^2_{t-1,j} \\
\varepsilon &\sim C\{F_1(x_1), \dots, F_d(x_d); \theta_t\}
\end{aligned}
$$

where $F_1, \dots, F_d$ are marginal distributions and θ_t are the copula parameters and $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $\alpha_j + \beta_j < 1$, $|\zeta| < 1$.

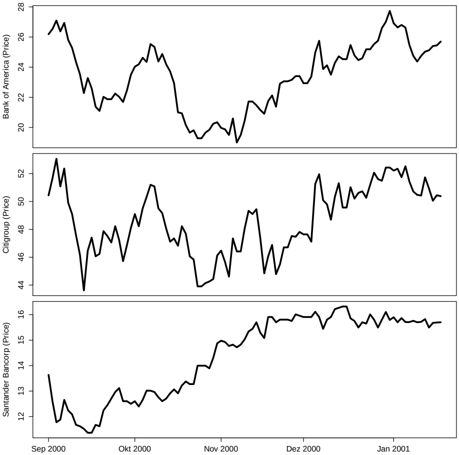


Figure 12: Stock prices for Bank of America, Citigroup and Santander (from top to bottom).

	$\widehat{\mu}_j$	$\widehat{\gamma}_j$	$\widehat{\zeta}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\beta}_j$
Bank of America	1.87e-3	0.22	-0.23	3.46e-4	0.55	0.17
(0.57, 0.83)	(2.59e-3)	(0.64)	(0.65)	(1.37e-04)	(0.28)	(0.16)
Citigroup	0.11e-3	0.31	-0.46	2.67e-4	0.09	0.47
(0.57, 0.79)	(1.48e-3)	(0.29)	(0.29)	(5.53e-04)	(0.17)	(1.01)
Santander	1.35e-3	0.43	-0.56	4.51e-10	0.01	0.98
(0.91, 0.78)	(0.91e-3)	(0.15)	(0.17)	(1.38e-05)	(0.02)	(0.05)

Table 10: Fitting of univariate ARMA(1,1)-GARCH(1,1) to asset returns. The standard deviation of the parameters, which are quiet big because of the small sample size, are given in parentheses. Each second row provides the p-values of the Box-Ljung test (BL) for autocorrelations and Kolmogorov-Smirnov test (KS) for testing of normality of the residuals.

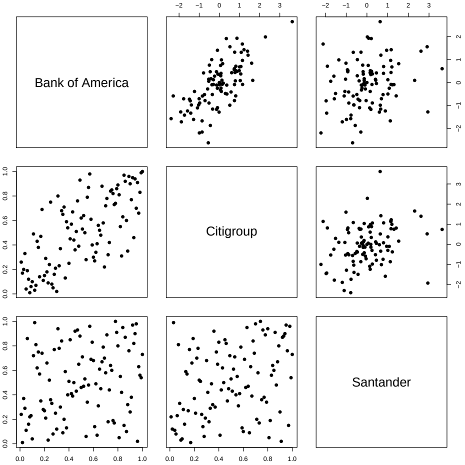


Figure 13: Scatterplots from ARMA-GARCH residuals (upper triangular) and from residuals mapped on unit square by the cdf (lower triangular).

	T_{100}	S_{100}	estimates
HAC	0.3191	0.1237	$C\{C(u_1, u_2; 1.996), u_3; 1.256\}$
AC	0.0012	0.0002	$C(u_1, u_2, u_3; 1.276)$
Gauss	0.0160	0.0078	$C_N\{u_1, u_2, u_3; \Sigma(0.697, 0.215, 0.312)\}$

Table 11: p-values of both GoFs and estimates of the models under different H_0 hypotheses.

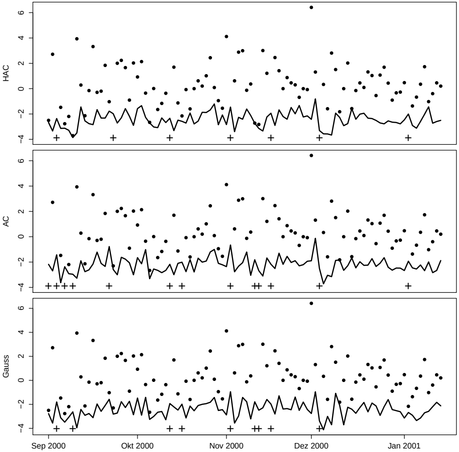


Figure 14: Profit and loss function and VaR based on different models.

α	$\widehat{\alpha}_{HAC}$	$\widehat{\alpha}_{AC}$	$\widehat{\alpha}_{Gauss}$
0.10	0.091	0.122	0.081
0.05	0.040	0.061	0.031
0.01	0.000	0.010	0.000

Table 12: Backtesting for the estimation of VaR under different alternatives.

Dependence orderings

C' is more **concordant** than C if
$(\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v))$

$$C \prec_c C' \Leftrightarrow C(\mathbf{x}) \leq C'(\mathbf{x}) \text{ and } \overline{C}(\mathbf{x}) \leq \overline{C'}(\mathbf{x}) \; \forall \mathbf{x} \in [0;1]^d.$$

Theorem

If two feasible hierarchical Archimedean copulas C^1 and C^2 differ only by the generator functions on the top level satisfying the condition $\phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$, then $C^1 \prec_c C^2$.

Theorem

If two hierarchical Archimedean copulas $C^1 = C^1_{\boldsymbol{\phi}_1}(u_1, \dots, u_d)$ and $C^2 = C^2_{\boldsymbol{\phi}_2}(u_1, \dots, u_d)$ differ only by the generator functions on the level r as $\boldsymbol{\phi}_1 = (\phi_1, \dots, \phi_{r-1}, \phi, \phi_{r+1}, \dots, \phi_p)$ and $\boldsymbol{\phi}_2 = (\phi_1, \dots, \phi_{r-1}, \phi^*, \phi_{r+1}, \dots, \phi_p)$ with $\phi^{-1} \circ \phi^* \in \mathcal{L}^*$, then $C^1 \prec_c C^2$.

Theorem

(Deheuvels (1978)) Let $\{X_{1i}, \dots, X_{di}\}_{i=1,\dots,n}$ *be a sequence of the random vectors with the distribution function* F*, marginal distributions* $F_1, \dots, F_d$ *and copula* C*. Let also* $M_j^{(n)} = \max_{1 \le i \le n} X_{ji}$*,* $j = 1, \dots, d$ *be the componentwise maxima. Then*

$$\lim_{n \to \infty} P \left\{ \frac{M_1^{(n)} - a_{1n}}{b_{1n}} \le x_1, \dots, \frac{M_d^{(n)} - a_{dn}}{b_{dn}} \le x_d \right\} = F^*(x_1, \dots, x_d), \quad \forall (x_1, \dots, x_d) \in \mathbb{R}^d$$

with $b_{jn} > 0,\ j = 1, \dots, d,\ n \ge 1$ *if and only if*

1. *for all* $j = 1, \dots, d$ *there exist some constants* a_{jn} *and* b_{jn} *and a non-degenerating limit distribution* F_j^* *such that*

$$\lim_{n \to \infty} P \left\{ \frac{M_j^{(n)} - a_{jn}}{b_{jn}} \le x_j \right\} = F_j^*(x_j), \quad \forall x_j \in \mathbb{R};$$

2. *there exists a copula* C^* *such that*

$$C^*(u_1, \dots, u_d) = \lim_{n \to \infty} C^n(u_1^{1/n}, \dots, u_d^{1/n}).$$

Let F_{ds} be the class of d dimensional hierarchical Archimedean copulas with structure s.

Theorem

If $C \in F_{ds_1}$, $C^* \in F_{ds_2}$, $\forall \varphi_\theta \in \mathcal{N}(C)$, $\partial[\varphi_\ell^{-1}(t)/(\varphi_\ell^{-1})'(t)]/\partial t\big|_{t=1}$ *exists and is equal to* $1/\theta$ *and* $C \in MDA(C^*)$ *and* $C \in MDA(C^*)$ *then* $s_1 = s_2$, $\forall \phi_\theta \in \mathbb{N}(C^*)$, $\phi_\theta(x) = \exp\{-x^{1/\theta}\}$.

If the multivariate HAC C (under some minor condition) belongs to the domain of attraction of the HAC C^**. The extreme value HAC* C^* *has the same structure as the given copula* C*, with generators on all levels of the hierarchy being Gumbel generators, but with probably other parameters.*

Tail dependency

The upper and lower tail indices of two random variables $X_1 \sim F_1$ and $X_2 \sim F_2$ are given by

$$\lambda_U = \lim_{u \to 1^-} P\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\} = \lim_{u \to 1^-} \frac{\overline{C}(u,u)}{1-u}$$

$$\lambda_L = \lim_{u \to 0^+} P\{X_2 \leq F_2^{-1}(u) \mid X_1 \leq F_1^{-1}(u)\} = \lim_{u \to 0^+} \frac{C(u,u)}{u}.$$

Theorem (Nelsen (1997))

For a bivariate Archimedean copula with the generator ϕ it holds

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - \phi\{2\phi^{-1}(u)\}}{1-u} = 2 - \lim_{w \to 0^+} \frac{1-\phi(2w)}{1-\phi(w)},$$

$$\lambda_L = \lim_{u \to 0^+} \frac{\phi\{2\phi^{-1}(u)\}}{u} = \lim_{w \to \infty} \frac{\phi(2w)}{\phi(w)}.$$

A function $\phi : (0, \infty) \to (0, \infty)$ is **regularly varying at infinity with tail index** $\lambda \in \boldsymbol{R}$ (written $RV_\lambda(\infty)$) if $\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = t^\lambda$ for all $t > 0$. $\phi \in RV_{-\infty}(\infty)$ if

$$\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = \begin{cases} \infty & \text{if} \quad t < 1 \\ 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t > 1 \end{cases} .$$

It holds for $\lambda \geq 0$ that if $\phi \in RV_{-\lambda}(\infty)$ then $\phi^{-1} \in RV_{-1/\lambda}(0)$. The function ϕ^{-1} is **regularly varying at zero with the tail index** γ, if $\lim_{w\to 0^+} \frac{\phi^{-1}(1-tw)}{\phi^{-1}(1-w)} = t^\gamma$. For the direct function $\lim_{w\to 0^+} \frac{1-\phi(tw)}{1-\phi(w)} = t^{1/\gamma}$.

$$\lim_{u\to 0^+} P\{X_i \le F_i^{-1}(u_i u) \ \text{for} \ i \notin \mathcal{S} \subset \mathcal{K} = \{1,\ldots,k\}$$
$$| \, X_j \le F_j^{-1}(u_j u) \ \text{for} \ j \in \mathcal{S}\}$$
$$\lim_{u\to 0^+} P\{X_i > F_i^{-1}(1 - u_i u) \ \text{for} \ i \notin \mathcal{S} \subset \mathcal{K} = \{1,\ldots,k\}$$
$$| \, X_j > F_j^{-1}(1 - u_j u) \ \text{for} \ j \in \mathcal{S}\}.$$

The above limits can be established via the limits

$$\lambda_L(u_1,\ldots,u_k) = \lim_{u\to 0^+} \frac{1}{u} C(u_1 u,\ldots,u_k u) \qquad \text{and}$$
$$\lambda_U(u_1,\ldots,u_k) = \lim_{u\to 0^+} \frac{1}{u} \overline{C}(1 - u_1 u,\ldots,1 - u_k u)$$
$$= \lim_{u\to 0^+} \sum_{s_1 \in \mathcal{K}} (-1)^{|s_1|+1} \{1 - C_{s_1}(1 - u_j u, j \in s_1)\}.$$

Theorem (Lower Tail Dependency)

Assume that the limits $\lim_{u\to 0^+} u^{-1} C_i(uu_{k_{i-1}+1}, \dots, uu_{k_i}) = \lambda_{L,i}(u_{k_{i-1}+1}, .., u_{k_i})$ *exist for all* $0 < u_{k_{i-1}+1}, .., u_{k_i} < 1,\ i = 1, \dots, m$. *Suppose that* $m + k - k_m \geq 2$. *If* ϕ_0^{-1} *is regularly varying at infinity with index* $-\lambda_0 \in [-\infty, 0]$, *then it holds for all* $0 < u_i < 1,\ i = 1, .., m$ *that*

$$
\lim_{u\to 0^+} \frac{C(uu_1, \dots, uu_k)}{u} = \begin{cases} \min\{\lambda_{L,1}(u_1, .., u_{k_1}), \dots, \lambda_{L,m}(u_{k_{m-1}+1}, .., u_{k_m}), u_{k_m+1}, \dots, u_k\} \\ \quad \textit{if} \quad \lambda_0 = \infty, \\ \left(\sum_{i=1}^{m} \lambda_{L,i}(u_{k_{i-1}+1}, .., u_{k_i})^{-\lambda_0} + \sum_{j=k_m+1}^{k} u_j^{-\lambda_0}\right)^{-1/\lambda_0} \\ \quad \textit{if} \quad 0 < \lambda_0 < \infty, \\ 0 \ \textit{if} \quad \lambda_0 = 0. \end{cases}
$$

In the following let

$$C_j^*(u) = C_j(u_{k_j-1+1}u, .., u_{k_j}u) \mid_{u_{k_{j-1}+1}=..=u_{k_j}=1},$$
$$C^*(u) = C(u_1 u, .., u_k u) \mid_{u_1=..=u_k=1},$$
$$\lambda_{L,j}^*(u, u_{k_{j-1}+1}, .., u_{k_j}) = C_j(u_{k_{j-1}+1}u, \dots, u_{k_j}u)/C_j^*(u).$$

Note that $0 \leq \lambda_{L,j}^*(u, u_{k_{j-1}+1}, .., u_{k_j}) \leq 1$. Moreover, if $\lim_{u\to 0^+} u^{-1} C_j(uu_{k_{j-1}+1}, \dots, uu_{k_j}) = \lambda_{L,j}(u_{k_{j-1}+1}, .., u_{k_j}) > 0$ for all $0 < u_{k_{j-1}+1}, .., u_{k_j} \leq 1$ then

$$\lambda_{L,j}^*(u_{k_{j-1}+1}, .., u_{k_j}) = \lim_{u\to 0+} \frac{C_j(u_{k_j-1+1}u, .., u_{k_j}u)/u}{C_j^*(u)/u}$$
$$= \frac{\lambda_{L,j}(u_{k_{j-1}+1}, .., u_{k_j})}{\lambda_{L,j}(1, .., 1)}$$

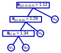
Theorem (Lower Tail Dependency 2)

Assume that the limits

$$\lim_{u\to 0^+} \frac{C_i(uu_{k_{i-1}+1},\ldots,uu_{k_i})}{C_i^*(u)} = \lambda^*_{L,i}(u_{k_{i-1}+1},..,u_{k_i})$$

exist for all $0 < u_{k_{i-1}+1},..,u_{k_i} \leq 1$, $i = 1,\ldots,m$. *Let* $\phi_0^{-1} \in RV_0(0)$ *and let* $\psi(v) = -\phi_0(v)/\phi_0'(v)$ *be regularly varying at infinity with finite tail index* $\varkappa$ *then* $\varkappa \leq 1$ *and it holds for all* $0 < u_i < 1$, $i = 1,..,m$ *that*

$$\lim_{u\to 0^+} \frac{C(uu_1,\ldots,uu_k)}{C^*(u)} = \prod_{j=1}^{m}[\lambda^*_{L,j}(u_{k_{j-1}+1},\ldots,u_{k_j})]^{(m+k-k_m)^{-\varkappa}} \cdot \prod_{j=k_m+1}^{k} u_j^{(m+k-k_m)^{-\varkappa}}.$$

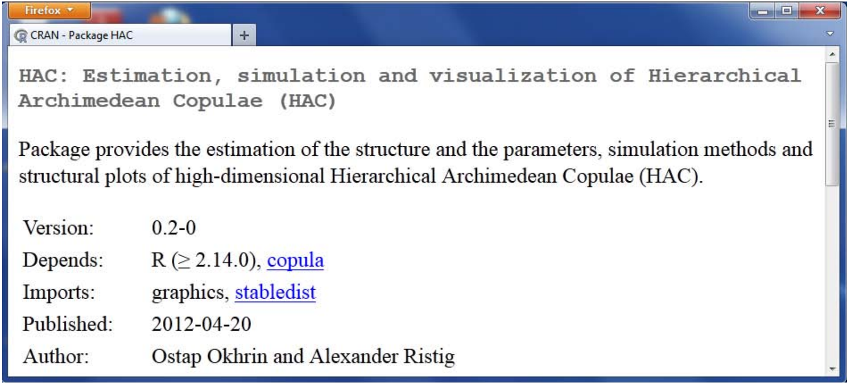


Theorem (Upper Tail Dependency)

Assume that the limits $\lim_{u\to 0^+} u^{-1}[1 - C_i(1 - uu_{k_{i-1}+1}, \ldots, 1 - uu_{k_i})] = \lambda_{U,i}(u_{k_{i-1}+1}, .., u_{k_i})$ *exist for all* $0 < u_{k_{i-1}+1}, .., u_{k_i} < 1$, $i = 1, \ldots, m$. *Suppose that* $m + k - k_m \geq 2$. *If* $\phi_0^{-1}(1 - w)$ *is regularly varying at zero with index* $-\gamma_0 \in [-\infty, -1]$, *then it holds for all* $0 < u_i < 1$, $i = 1, .., m$ *that*

$$\lim_{u\to 0^+} \frac{1 - C(1 - uu_1, \ldots, 1 - uu_k)}{u} = \begin{cases} \min\{\lambda_{U,1}(u_1, .., u_{k_1}), \ldots, \lambda_{U,m}(u_{k_{m-1}+1}, .., u_{k_m}), u_{k_m+1}, \ldots, u_k\} \\ \qquad \text{if} \quad \gamma_0 = \infty, \\ \left(\sum_{i=1}^m [\lambda_{U,i}(u_{k_{i-1}+1}, .., u_{k_i})]^{\gamma_0} + \sum_{j=k_m+1}^k u_j^{\gamma_0}\right)^{1/\gamma_0} \\ \qquad \text{if} \quad 1 \leq \gamma_0 < \infty, \end{cases}$$

HAC meets R



HAC: Estimation, simulation and visualization of Hierarchical Archimedean Copulae (HAC)

Package provides the estimation of the structure and the parameters, simulation methods and structural plots of high-dimensional Hierarchical Archimedean Copulae (HAC).

Version:	0.2-0
Depends:	R ($\geq$ 2.14.0), copula
Imports:	graphics, stabledist
Published:	2012-04-20
Author:	Ostap Okhrin and Alexander Ristig

Figure 15: Website for downloading

Portfolio Management

- HAC can be applied to VaR estimation or assessing diversification effects.
- Four stocks: CVX, FP, RDSA and XOM.
- 20110202 to 20120319

```
> price = read.table("stocks")
> ret = diff(log(price), 1)
```

- Residuals of ARMA-GARCH models `res`
- Non-ellipticity? Joint extreme events?

```
> pairs(ret, pch = 20)
```

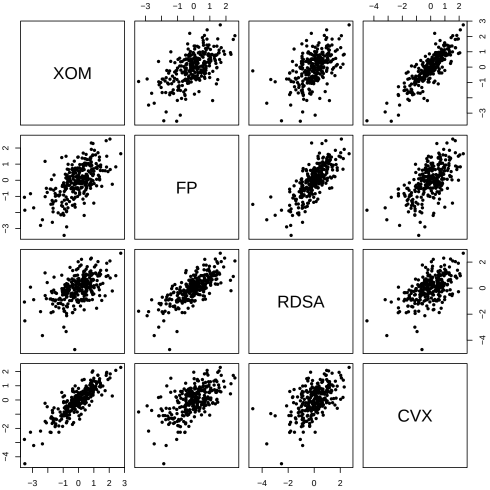


Figure 16: Dependencies of CVX, FP, RDSA and XOM

- Copula estimation based on uniformly distributed margins ures

```
> result = estimate.copula(ures)
> plot(result)
```

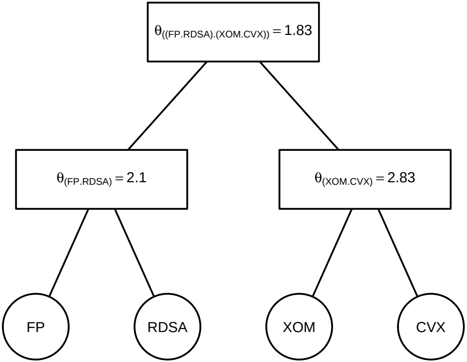


Figure 17: Estimated HAC of the portfolio

Estimation

- 3 computational blocks:
  1. Specification of the margins
  2. Estimation of the parameters and the structure
  3. Optional aggregation of the binary HAC
- Two estimation procedures: QML and Kendall's τ.
- `estimate.copula` returns a `hac` object.

```r
> result1 = estimate.copula(res, margins = 'edf')
> plot(result1)
```

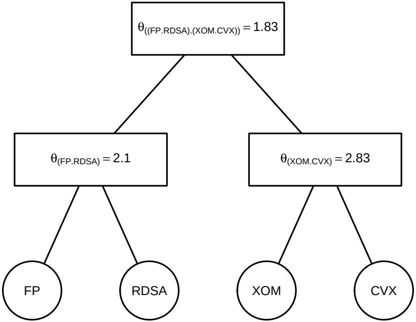


Figure 18: Estimation result

- Note, $C_{\theta_{1(23)}}\left(C_{\theta_{23}}(u_2,u_3),u_1\right) = C_{\theta_{123}}(u_1,u_2,u_3)$, if $|\theta_{1(23)} - \theta_{23}| < \varepsilon, \varepsilon > 0$

- `epsilon = 0.3` leads to a non-binary structure

```
> result2 = estimate.copula(X = res,
+       type = HAC_GUMBEL, method = ML,
+       epsilon = 0.3, agg.method = "mean"
+       margins = "edf")
> plot(result2)
```

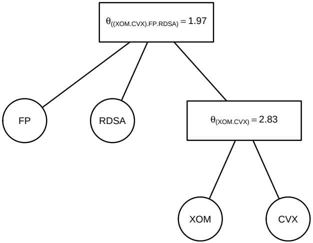


Figure 19: Results of the modified estimation

Objects of the class hac

- hac and hac.full create objects of the class hac.
- hac.full cannot construct partially nested AC.
- Consider a 5-dimensional fully nested Gumbel HAC:

```
> G1 = hac.full(type = HAC_GUMBEL,
+       y = c("X1", "X2", "X3", "X4", "X5"),
+       theta = c(1, 1.01, 2, 2.01))
> G1
Class: hac
Generator: Gumbel
((((X5.X4)_{2.01}.X3)_{2}.X2)_{1.01}.X1)_{1}
```

- It is smarter to aggregate the variables X1 and X2 in a first node and the variables X3, X4 and X5 in a second node.

```
> G2 = hac(type = HAC_GUMBEL,
+       tree = list(list("X3", "X4", "X5", 2.005),
+       "X2", "X1", 1.005))
```

- Substituting of variables for lists leads to arbitrary objects

```
> G3 = hac(tree = list(list("Y1", "Y2",
+       list("Z3", "Z4", 3), 2.5),
+       list("Z1", "Z2", 2),
+       list("X1", "X2", 2.4),
+       "X3", "X4", 1.5))
```

Graphics

```
> plot(G3)
```

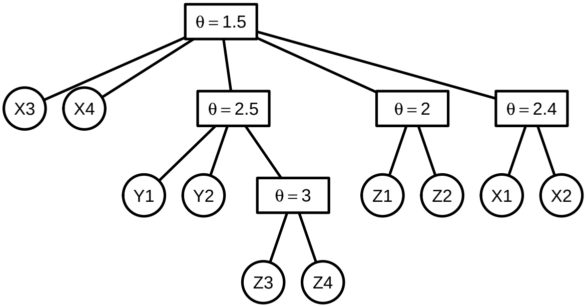


Figure 20: Structure of object G3

```
> plot(G3, digits = 2, theta = TRUE,
+       col = "blue3", fg = "red3",
+       bg = "white", col.t = "blue3")
```

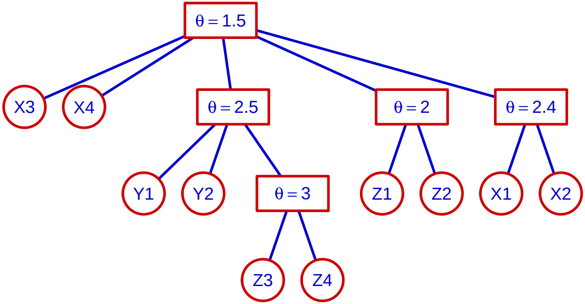


Figure 21: Colored structure of object G3

```
> tree2str(hac = G2, theta = TRUE
+          digits = 3)
[1] ``((X3.X4.X5)_{2.005}.X2.X1)_{1.005}''
> plot(G2, digits = 3, index = TRUE,
+      theta = FALSE)
```

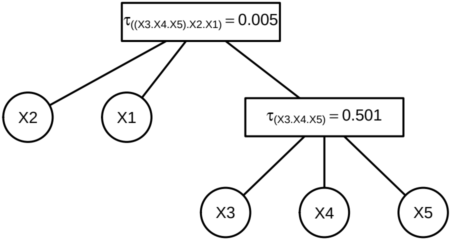


Figure 22: Structure of object G2

Simulation

- Simulation of HAC requires 2 arguments: the number of generated random vectors and a `hac` object.

```
> sample = rHAC(n = 1500, hac = G2)
```

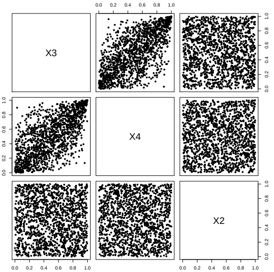


Figure 23: Scatterplot of `sample`

Distribution Functions

- pHAC computes the values of copulae.

```
> cf.values = pHAC(X = sample, hac = G2)
```

- emp.copula.self computes the empirical copula, i.e.
  $\widehat{C}(u_1, \ldots, u_d) = n^{-1} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathrm{I}\left\{\widehat{F}_j(X_{ij}) \leq u_j\right\}.$

```
> ecf.values = emp.copula.self(x = sample,
+   proc = "M", sort = "none", na.rm = FALSE)
```

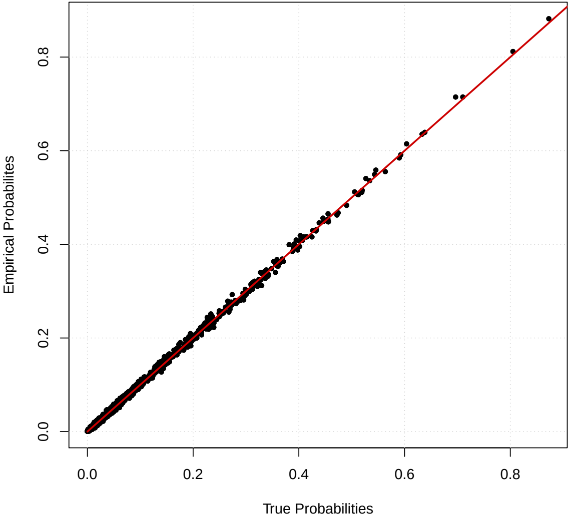


Figure 24: Values of `cf.values` on the x-axis against the values of the `ecf.values`

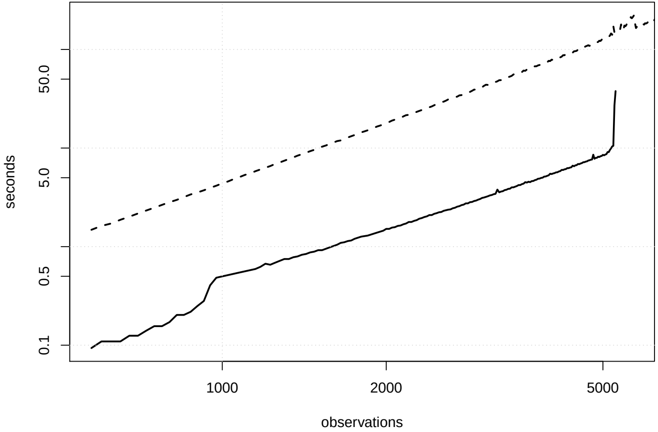


Figure 25: Runtimes of `emp.copula.self` for an increasing sample-size but fixed dimension $d = 5$ plotted on a log-log-scale

Density Functions

- d-dimensional copula density

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \cdots \partial u_d}.$$

- `dHAC` returns the values of the analytical density.
  - Requires a data matrix and a `hac` object as arguments.
- Construction of Likelihood functions by `to.logLik`.
- Random sampling using conditional inverse method.

Local Change Point Detection

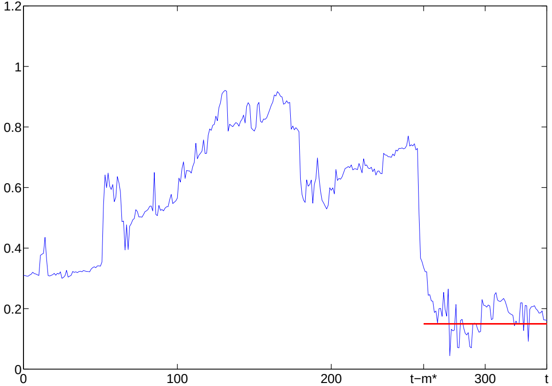


Figure 26: Dependence over time for DaimlerChrysler, Volkswagen, Bayer, BASF, Allianz and Münchener Rückversicherung, 20000101-20041231. Giacomini et. al (2009)

Adaptive Copula Estimation

- adaptively estimate largest interval where homogeneity hypothesis is accepted
- *Local Change Point* detection (LCP): sequentially test $\theta_t, \; s_t$ are constants (i.e. $\theta_t = \theta, \; s_t = s$) within some interval I (local parametric assumption).

- "*Oracle*" choice: largest interval $I = [t_0 - m_{k^*}, t_0]$ where small modelling bias condition (SMB)

$$\triangle_I(s, \boldsymbol{\theta}) = \sum_{t \in I} \mathcal{K}\{C(\cdot; s_t, \boldsymbol{\theta}_t), C(\cdot; s, \boldsymbol{\theta})\} \leq \triangle.$$

holds for some $\triangle \geq 0$. m_{k^*} is the ideal scale, $(s,\ \theta)^\top$ is ideally estimated from $I = [t_0 - m_{k^*}, t_0]$ and $\mathcal{K}(\cdot, \cdot)$ is the *Kullback-Leibler* divergence

$$\mathcal{K}\{C(\cdot; s_t, \boldsymbol{\theta}_t), C(\cdot; s, \boldsymbol{\theta})\} = \boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \frac{c(\cdot; s_t, \boldsymbol{\theta}_t)}{c(\cdot; s, \boldsymbol{\theta})}$$

Under the SMB condition on I_{k^*} and assuming that $\max_{k \le k^*} \boldsymbol{E}_{s,\boldsymbol{\theta}} |\mathcal{L}(\widetilde{s}_k, \widetilde{\boldsymbol{\theta}}_k) - \mathcal{L}(s, \boldsymbol{\theta})|^r \le \mathcal{R}_r(s, \boldsymbol{\theta})$, we obtain

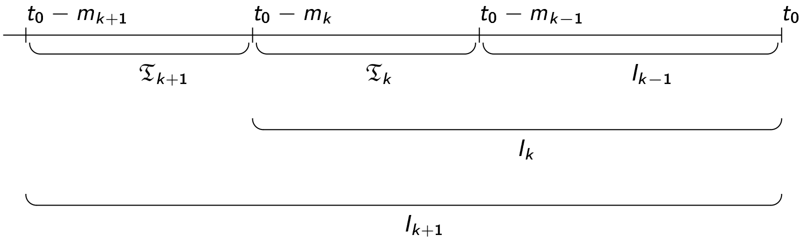
$$\boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{s}_{\widehat{k}}, \widetilde{\boldsymbol{\theta}}_{\widehat{k}}) - \mathcal{L}(s, \boldsymbol{\theta})|^r}{\mathcal{R}_r(s, \boldsymbol{\theta})} \right\} \le 1 + \triangle,$$

$$\boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{s}_{\widehat{k}}, \widetilde{\boldsymbol{\theta}}_{\widehat{k}}) - \mathcal{L}(\widehat{s}_{\widehat{k}}, \widehat{\boldsymbol{\theta}}_{\widehat{k}})|^r}{\mathcal{R}_r(s, \boldsymbol{\theta})} \right\} \le \rho + \triangle,$$

where $\widehat{a}_I$ is an adaptive estimator on I and $\widetilde{a}_I$ is any other parametric estimator on I.

Local Change Point Detection

1. define family of nested intervals
   $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ with length m_k as
   $I_k = [t_0 - m_k, t_0]$
2. define $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$



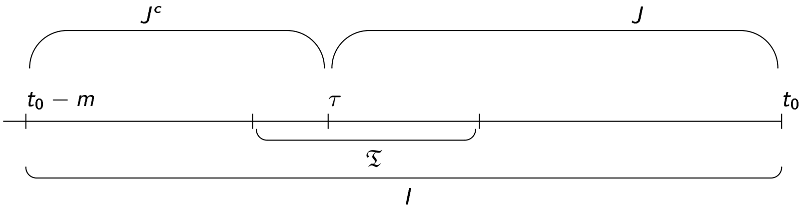
Local Change Point Detection

1. test homogeneity $H_{0,k}$ against the change point alternative in $\mathfrak{T}_k$ using I_{k+1}
2. if no change points in $\mathfrak{T}_k$, accept I_k. Take $\mathfrak{T}_{k+1}$ and repeat previous step until $H_{0,k}$ is rejected or largest possible interval I_K is accepted
3. if $H_{0,k}$ is rejected in $\mathfrak{T}_k$, homogeneity interval is the last accepted $\widehat{I} = I_{k-1}$
4. if largest possible interval I_K is accepted $\widehat{I} = I_K$

Test of homogeneity

Interval $I = [t_0 - m, t_0], \mathfrak{T} \subset I$

$$
\begin{aligned}
H_0 \quad &: \quad \forall \tau \in \mathfrak{T},\ \theta_t = \theta,\ s_t = s, \\
&\quad\ \ \forall t \in J = [\tau, t_0], \forall t \in J^c = I \setminus J \\
H_1 \quad &: \quad \exists \tau \in \mathfrak{T},\ \theta_t = \theta_1,\ s_t = s_1;\ \forall t \in J, \\
&\quad\ \ \theta_t = \theta_2 \neq \theta_1;\ s_t = s_2 \neq s_1, \forall t \in J^c
\end{aligned}
$$



Test of homogeneity

Likelihood ratio test statistic for fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2}\{L_J(\theta_1)+L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= ML_J + ML_{J^c} - ML_I \end{aligned}$$

Test statistic for unknown change point location:

$$T_I = \max_{\tau \in \mathfrak{T}_I} T_{I,\tau}$$

Reject H_0 if for a critical value ζ_I

$$T_I > \zeta_I$$

Selection of I_k and ζ_k

- set of numbers m_k defining the length of I_k and $\mathfrak{T}_k$ are in the form of a geometric grid
- $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 \in \{20,\ 40\}$, $c = 1.25$ and $K = 10$, where $[x]$ means the integer part of x
- $I_k = [t_0 - m_k, t_0]$ and $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$

(Mystery Parameters)

Sequential choice of ζ_k

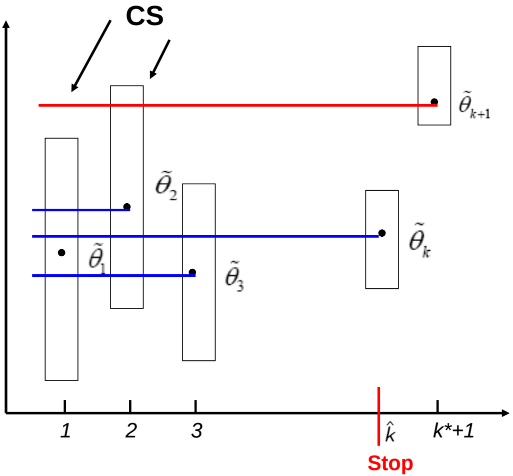
- after k steps are two cases: change point at some step $\ell \le k$ and no change points.
- let $\mathcal{B}_\ell$ be the event meaning the rejection at step ℓ

$$\mathcal{B}_\ell = \{T_1 \le \zeta_1, \dots, T_{\ell-1} \le \zeta_{\ell-1}, T_\ell > \zeta_\ell\},$$

  and $(\widehat{s}_k, \widehat{\boldsymbol{\theta}}_k) = (\widetilde{s}_{\ell-1}, \widetilde{\boldsymbol{\theta}}_{\ell-1})$ on $\mathcal{B}_\ell$ for $\ell = 1, \dots, k$.
- we find sequentially such a minimal value of ζ_ℓ that ensures following inequality

$$\max_{k=l,\dots,K} \boldsymbol{E}_{s^*,\theta^*} |\mathcal{L}(\widetilde{s}_k, \widetilde{\boldsymbol{\theta}}_k) - \mathcal{L}(\widetilde{s}_{\ell-1}, \widetilde{\boldsymbol{\theta}}_{\ell-1})|^r \mathbf{I}(\mathcal{B}_\ell) \le \rho \mathcal{R}_r(s^*, \boldsymbol{\theta}^*) \frac{k}{K-1}$$

Illustration



Sequential choice of ζ_k

1. pairs of Kendall's τ: $\forall \{\tau_1, \tau_2\} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}^2,\ \tau_1 \geq \tau_2$
2. simul. from $C_{\theta_i,\theta_j}(u_1, u_2, u_3) = C\{C(u_1, u_2; \theta_1), u_3; \theta_2\},\ \theta = \theta(\tau)$
3. run sequential algorithm for each sample

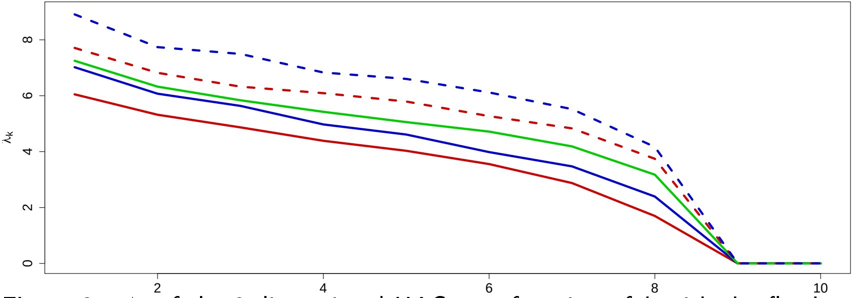


Figure 27: ζ_k of the 3-dimensional HAC as a function of k with the fixed $m_0 = 40$, $\rho = 0.5$, $r = 0.5$, $\tau_1 = 0.1$ and for different τ_2. $\tau_2 = 0.1$ (solid), $\tau_2 = 0.3$ (solid), $\tau_2 = 0.5$ (solid), $\tau_2 = 0.7$ (dashed), $\tau_2 = 0.9$ (dashed).

Simulation: Change in θ_1, I

$$C_t(u_1, u_2, u_3; s, \theta) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 3.33); \theta_2 = 1.43\} & \text{for } 1 \le t \le 200 \\ C\{u_1, C(u_2, u_3; \theta_1 = 2.00); \theta_2 = 1.43\} & \text{for } 200 < t \le 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

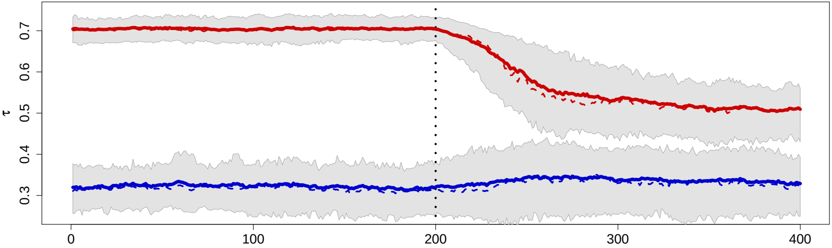


Figure 28: θ_1 and θ_2 on the intervals of homogeneity (median - dashed line, mean - solid line).

Simulation: Change in θ_1, II

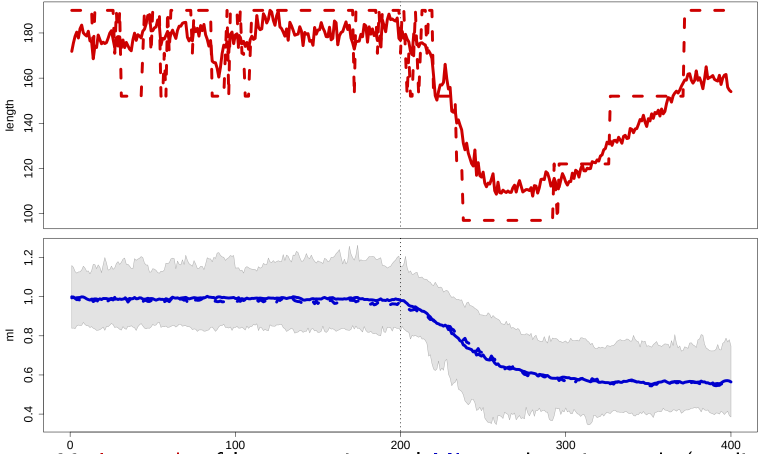


Figure 29: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Simulation: Change in θ_2, I

$$C_t(u_1,u_2,u_3;s,\boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2,u_3;\theta_1 = 3.33);\theta_2 = 1.43\} & \text{for } 1 \leq t \leq 200 \\ C\{u_1, C(u_2,u_3;\theta_1 = 3.33);\theta_2 = 2.00\} & \text{for } 200 < t \leq 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

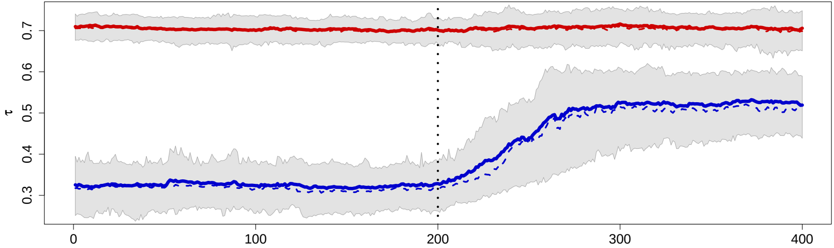


Figure 30: θ_1 and θ_2 on the intervals of homogeneity (median - dashed line, mean - solid line).

Simulation: Change in θ_2, II

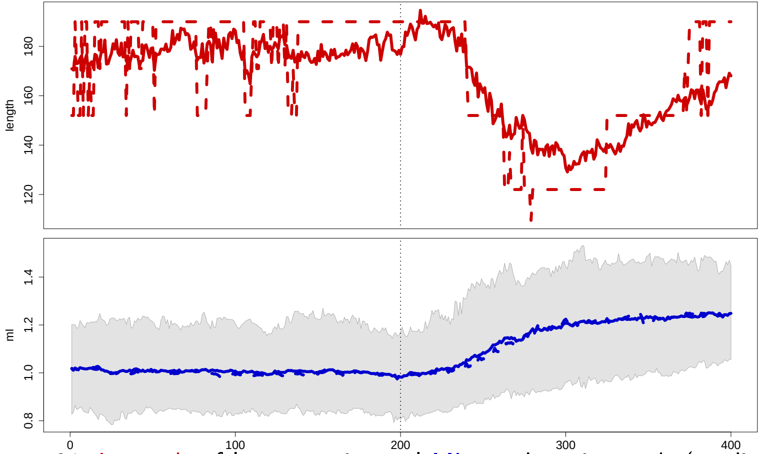


Figure 31: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Simulation: Change in the Structure, I

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 3.33); \theta_2 = 1.43\} & \text{for } 1 \leq t \leq 200 \\ C\{C(u_1, u_2; \theta_1 = 3.33), u_3; \theta_2 = 1.43\} & \text{for } 200 < t \leq 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

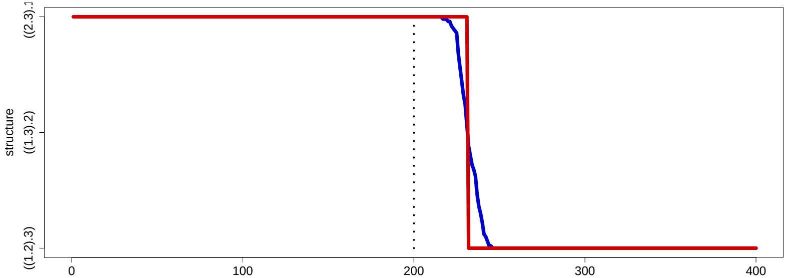


Figure 32: The structure of the est. HAC on the intervals of homogeneity (median - dashed line, mean - solid line)

Simulation: Change in the Structure, II

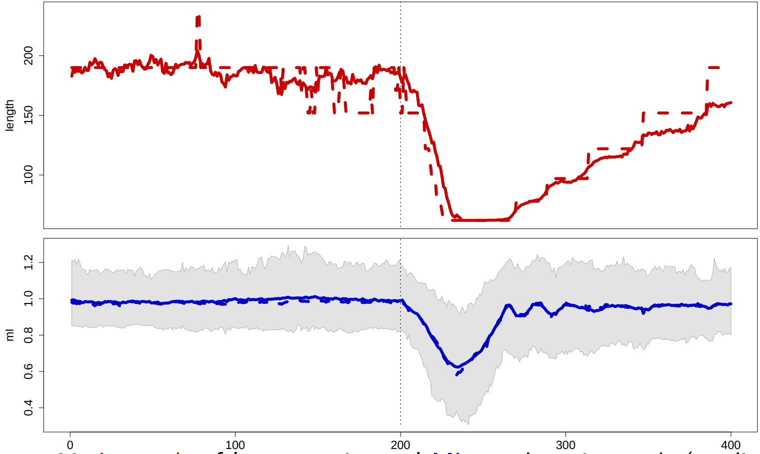


Figure 33: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Data and Copula

- daily values for the exchange rates
  JPN/USD, GBP/USD and EUR/USD
- timespan = [4.1.1999; 14.8.2009] ($n = 2771$)
- Gumbel and Clayton generators generators

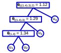
Data and Copula

- a univariate GARCH(1,1) process on log-returns

$$
\begin{aligned}
X_{j,t} &= \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t} \text{ with } \sigma^2_{j,t} = \omega_j + \alpha_j \sigma^2_{j,t-1} + \beta_j (X_{j,t-1} - \mu_{j,t-1})^2 \\
\varepsilon_t &\sim C\{F_1(x_1), \ldots, F_d(x_d); \theta_t\}
\end{aligned}
$$

	$\widehat{\mu}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\beta}_j$	BL	KS
JPY	4.85e-05	2.99e-07	0.06	0.94	0.73	1.70e-05
	(1.15e-04)	(1.02e-07)	(7.49e-03)	(7.64e-03)		
GBP	6.34e-05	1.44e-07	0.06	0.93	0.01	2.10e-04
	(7.39e-05)	(5.11e-08)	(8.75e-03)	(9.12e-03)		
USD	1.76e-04	1.19e-07	0.03	0.97	0.87	1.65e-03
	(1.10e-04)	(5.92e-08)	(4.14e-03)	(4.28e-03)		

Table 13: Estimation results univariate time series modelling.



HAC for whole sample

Generator	Structure	ML
Clayton	$((\text{JPY.USD})_{0.808(0.042)}.\text{GBP})_{0.401(0.025)}$	617.268
Gumbel	$((\text{JPY.USD})_{1.521(0.025)}.\text{GBP})_{1.303(0.016)}$	736.341

Table 14: Estimation results for HAC with Clayton and Gumbel generators for indices and exchange rates using full samples. The standard deviations of the parameters are given in the parenthesis.

LCP for HAC to real Data

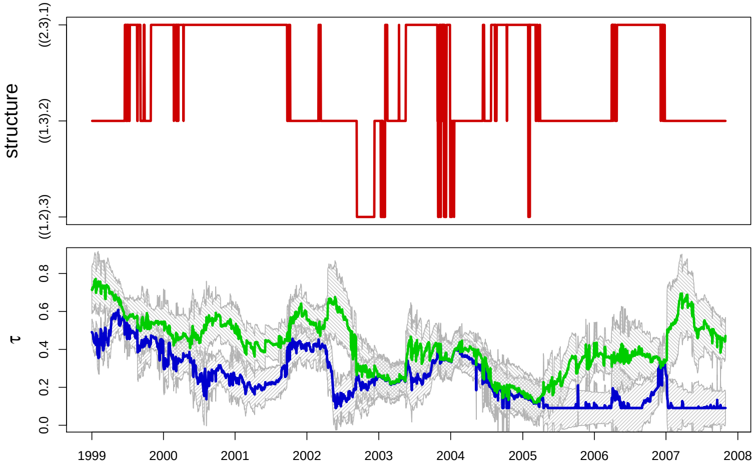


Figure 34: Structure, τ_1 and τ_2 of the HAC on the intervals of homogeneity

LCP for HAC to real Data

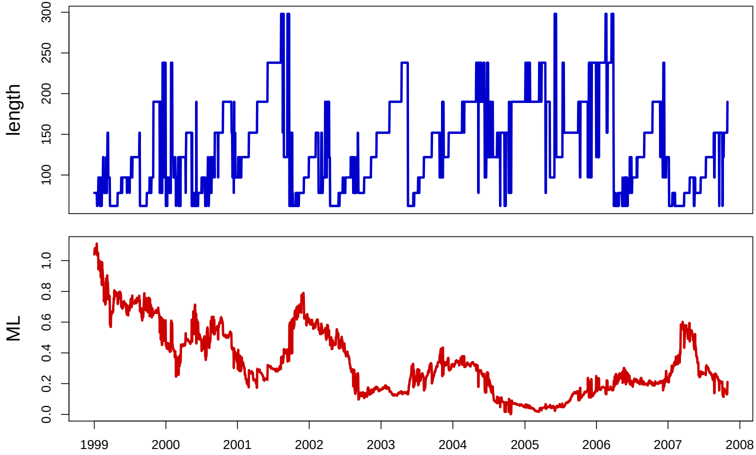


Figure 35: Intervals of homogeneity and ML on these intervals

VaR

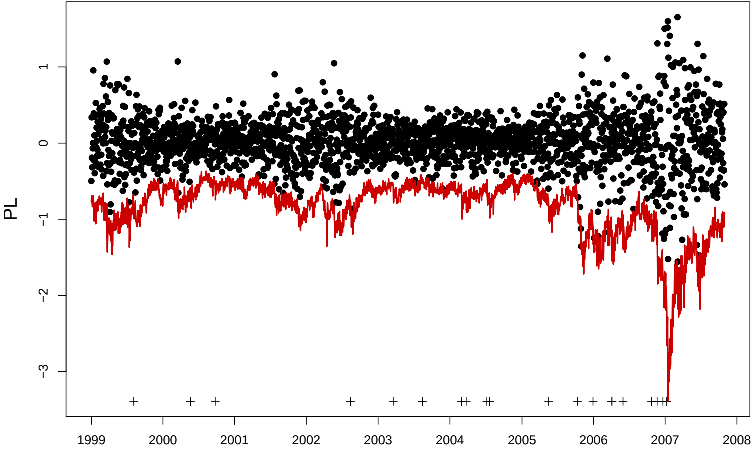


Figure 36: Profit and Loss function

VaR

α	Clayton			Gumbel			DCC		
α	0.0100	0.0500	0.1000	0.0100	0.0500	0.1000	0.0100	0.0500	0.1000
$\widehat{\alpha}_{w^*}$	0.0100	0.0487	0.0951	0.0091	0.0474	0.0977	0.0156	0.0413	0.0817
$\widehat{\alpha}_{w_1}$	0.0083	0.0460	0.0912	0.0087	0.0447	0.0925	0.0152	0.0408	0.0812
$\widehat{\alpha}_{w_2}$	0.0100	0.0487	0.0951	0.0096	0.0487	0.0977	0.0156	0.0413	0.0812
$\widehat{\alpha}_{w_3}$	0.0100	0.0487	0.0951	0.0091	0.0482	0.0973	0.0156	0.0413	0.0812
$\widehat{\alpha}_{w_4}$	0.0100	0.0487	0.0951	0.0091	0.0469	0.0973	0.0156	0.0417	0.0817
$\widehat{\alpha}_{w_5}$	0.0100	0.0487	0.0947	0.0091	0.0478	0.0973	0.0156	0.0417	0.0817
A_W	-0.0217	-0.0328	-0.0557	-0.0895	-0.0526	-0.0341	0.5482	-0.1652	-0.1852
D_W	0.0649	0.0186	0.0125	0.0632	0.0406	0.0272	0.0335	0.0091	0.0042

Table 15: Exceedance ratios for portfolios of exchange rates with w^*, $w_i, i = 1, \dots, 5$, the average exceedance A_W over all portfolios and its standard deviation D_W.

Data and Copula

- daily returns values for Dow Jones (DJ), DAX and NIKKEI
- timespan = [01.01.1985; 23.12.2010] ($n = 6778$)
- Gumbel and Clayton generators

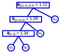
APARCH(1,1) model with the residuals following the skewed-t distribution

$$
\begin{aligned}
X_{j,t} &= \mu_j + \sigma_{j,t}\varepsilon_{j,t} \\
\text{with} \quad & \sigma_{j,t}^{\delta_j} = \omega_j + \alpha_j(|X_{j,t-1} - \mu_j| - \gamma(X_{j,t-1} - \mu_j))^{\delta_j} + \beta_j \sigma_{j,t-1}^{\delta_j},
\end{aligned}
$$

where $\varepsilon_{j,t} \sim t_{\text{skewed}}(\varkappa;\nu)$, $j = 1, \ldots, 3$. The parameters $\varkappa$ and ν stand for the skew and shape (degrees of freedom) of the distribution.

	$\widehat{\mu}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\gamma}_j$	$\widehat{\beta}_j$	$\widehat{\delta}_j$	$\widehat{\varkappa}_j$	$\widehat{\nu}_j$	BL	KS
DAX	4.828e-04	5.797e-05	8.495e-02	0.428	9.120e-01	1.305	9.244e-01	8.104	0.113	2.790e-07
	(1.261e-04)	(9.094e-06)	(7.290e-03)	(0.058)	(7.191e-03)	(0.110)	(1.587e-02)	(0.700)		
DJ	4.015e-04	8.077e-05	6.141e-02	0.659	9.382e-01	1.118	9.573e-01	5.587	0.277	9.259e-14
	(9.772e-05)	(1.367e-05)	(5.961e-03)	(0.088)	(5.467e-03)	(0.113)	(1.502e-02)	(0.392)		
NIKKEI	1.614e-04	4.940e-05	8.389e-02	0.509	9.180e-01	1.299	9.556e-01	6.253	0.198	9.259e-12
	(1.249e-04)	(8.148e-06)	(6.961e-03)	(0.063)	(6.238e-03)	(0.113)	(1.507e-02)	(0.477)		

Table 16: Estimation results univariate time series modelling.



HAC for whole sample

Generator	Structure	ML
Clayton	$((\text{DAX.DJ})_{0.459(0.021)}.\text{NIKKEI})_{0.155(0.012)}$	545.399
Gumbel	$((\text{DAX.DJ})_{1.272(0.012)}.\text{NIKKEI})_{1.103(0.007)}$	542.736

Table 17: Estimation results for HAC with Clayton and Gumbel generators for indices and exchange rates using full samples. The standard deviations of the parameters are given in the parenthesis.

Copulae over time

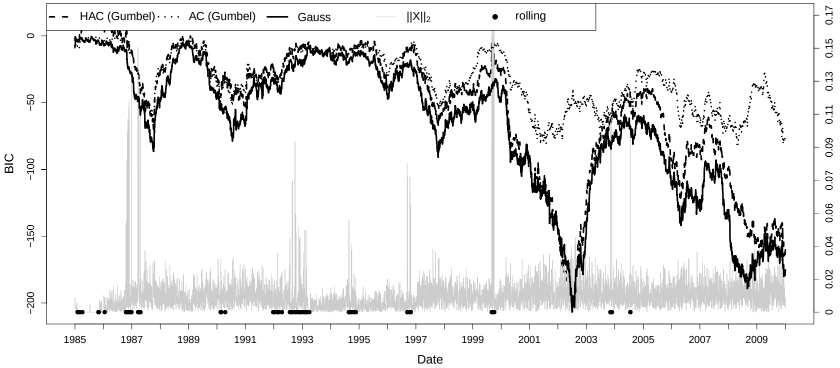


Figure 37: Time-varying HAC: BIC for the AC, Gaussian copula and HAC. Difference Matrix and points of the changes of the structure.

LCP for HAC to real Data

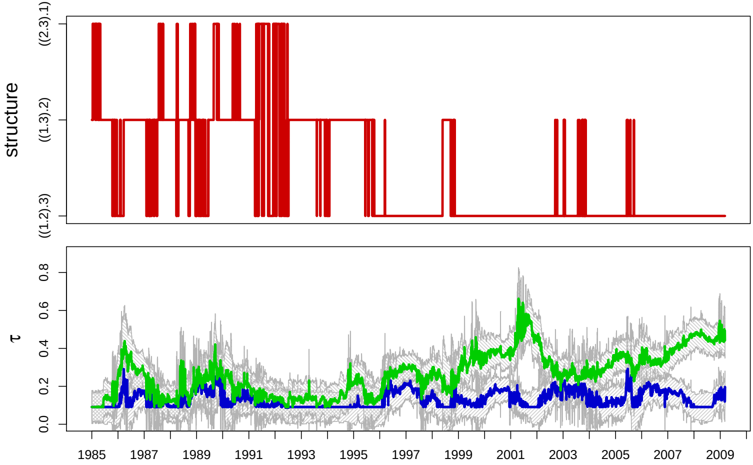


Figure 38: Structure, τ_1 and τ_2 of the HAC on the intervals of homogeneity

LCP for HAC to real Data

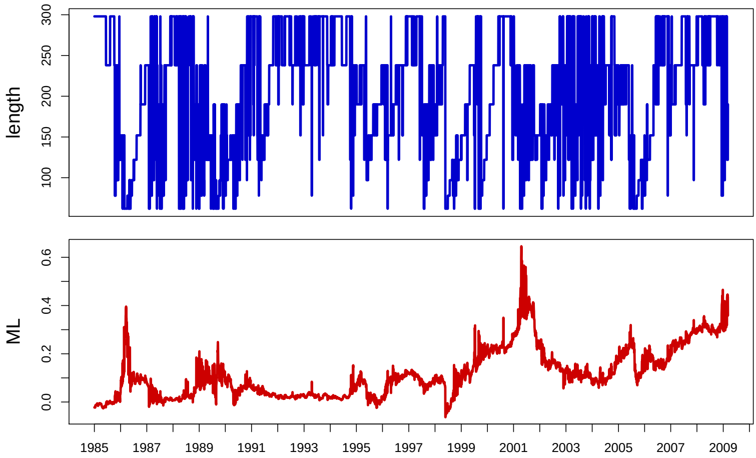


Figure 39: Intervals of homogeneity and ML on these intervals

VaR

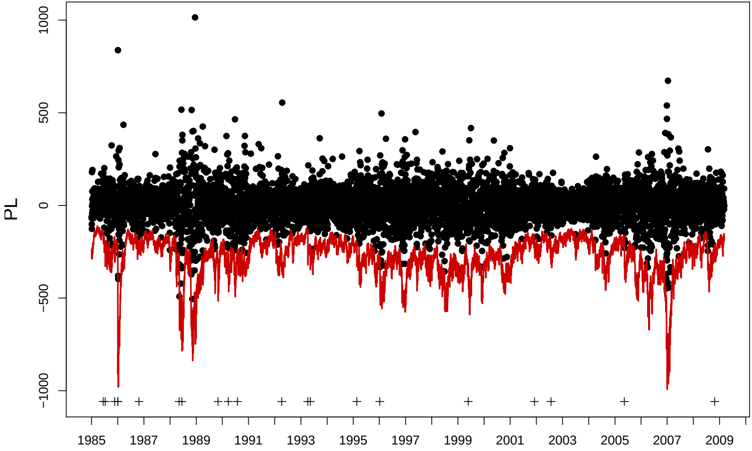


Figure 40: Profit and Loss function

VaR

	Clayton			Gumbel			DCC		
α	0.0100	0.0500	0.1000	0.0100	0.0500	0.1000	0.0100	0.0500	0.1000
$\widehat{\alpha}_{w^*}$	0.0054	0.0390	0.0935	0.0033	0.0249	0.0683	0.0155	0.0460	0.0830
$\widehat{\alpha}_{w_1}$	0.0055	0.0372	0.0916	0.0040	0.0239	0.0705	0.0162	0.0453	0.0864
$\widehat{\alpha}_{w_2}$	0.0073	0.0458	0.0994	0.0044	0.0303	0.0788	0.0152	0.0471	0.0830
$\widehat{\alpha}_{w_3}$	0.0055	0.0412	0.0940	0.0030	0.0254	0.0718	0.0160	0.0480	0.0808
$\widehat{\alpha}_{w_4}$	0.0052	0.0399	0.0943	0.0035	0.0225	0.0681	0.0157	0.0431	0.0818
$\widehat{\alpha}_{w_5}$	0.0062	0.0422	0.0976	0.0043	0.0290	0.0765	0.0160	0.0507	0.0887
A_W	-0.3902	-0.1781	-0.0497	-0.6187	-0.4496	-0.2686	0.5979	-0.0687	-0.1739
D_W	0.0930	0.0508	0.0286	0.0953	0.0932	0.0638	0.0959	0.0829	0.0609

Table 18: Exceedance ratios for portfolios of indices with w^*, $w_i, i = 1, \ldots, 5$, the average exceedance A_W over all portfolios and its standard deviation D_W.

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