Fitting copula to Data

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Outline

1. Motivation ✓
2. Hierarchical Archimedean copulae
3. Recovering the Structure
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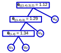
Gaussian Copula

$$
\begin{aligned}
C^G_\delta(u_1,u_2) &= \Phi_\delta\{\Phi^{-1}(u_1),\Phi^{-1}(u_2)\}\\
&= \int\limits_{-\infty}^{\Phi^{-1}(u_1)}\int\limits_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{\frac{-(s^2-2\delta st+t^2)}{2(1-\delta^2)}\right\}ds\,dt,
\end{aligned}
$$

- Gaussian copula contains the dependence structure
- *normal* marginal distribution + Gaussian copula = *multivariate normal distributions*
- *non-normal* marginal distribution + Gaussian copula = *meta-Gaussian distributions*
- allows to generate joint symmetric dependence, but **no** tail dependence



Archimedean Copula

Multivariate Archimedean copula $C : [0,1]^d \to [0,1]$ defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \tag{1}$$

where $\phi : [0,\infty) \to [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.

Example

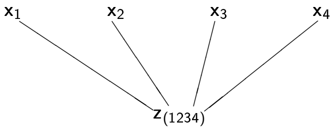
$$\begin{aligned} \phi_{Gumbel}(u,\theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \le \theta < \infty \\ \phi_{Clayton}(u,\theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1,\infty) \backslash \{0\} \end{aligned}$$

Disadvantages: too restrictive, single parameter, exchangeable

Hierarchical Archimedean Copulae

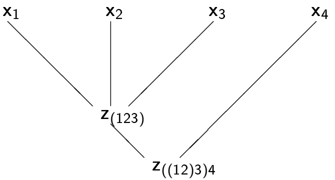
Simple AC with s=(1234)

$$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$$



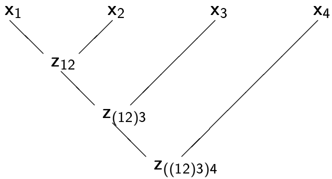
AC with s=((123)4)

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$$



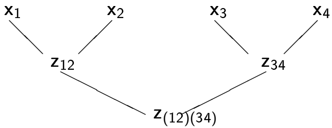
Fully nested AC with s=(((12)3)4)

$$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$$



Partially Nested AC with s=((12)(34))

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$$



Theoretical motivation

Let M be the cdf of a positive random variable and ϕ denotes its Laplace transform, i.e. $\phi(t) = \int_0^\infty e^{-tw} dM(w)$. For an arbitrary cdf F there exists a unique cdf G, such that

$$F(x) = \int_0^\infty G^\alpha(x) dM(\alpha) = \phi\{-\ln G(x)\}.$$

Now consider a k-variate cumulative distribution function F with margins $F_1, \ldots, F_d$. Then it holds for $G_j = \exp\{-\phi^{-1}(F_j)\}$ that

$$\int_0^\infty G_1^\alpha(x_1) \cdot \cdots \cdot G_d^\alpha(x_d) dM(\alpha) = \phi\left\{-\sum_{i=1}^d \ln G_i(x_i)\right\} = \phi\left[\sum_{i=1}^d \phi^{-1}\{F_i(x_i)\}\right].$$

$$C(u_1, \ldots, u_d) =$$

$$\int_0^\infty \ldots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1, \alpha_2)\; G_3^{\alpha_2}(u_3) dM_2(\alpha_2, \alpha_3) \ldots G_d^{\alpha_{d-1}}(u_d) dM_{d-1}(\alpha_{d-1}).$$

Recovering the structure (theory)

To guarantee that C is a HAC we assume that $\phi^{-1}_{d-i} \circ \phi_{d-j} \in \mathcal{L}^*$, $i < j$ with

$$\mathcal{L}^* = \{\omega : [0,\infty) \to [0,\infty) \,|\, \omega(0) = 0,\ \omega(\infty) = \infty,\ (-1)^{j-1}\omega^{(j)} \geq 0,\ j \geq 1\}.$$

☞ for most of the generator functions the parameters should decrease from the lowest level to the highest

Theorem

Let F be an arbitrary multivariate distribution function based on HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.

Estimation Issues

$$
\begin{aligned}
F_j(x;\widehat{\alpha}_j) &= F_j(x; arg\max_\alpha \sum_{i=1}^{n} \log\ f_j(X_{ji},\alpha)),\\
\widehat{F}_j(x) &= \frac{1}{n+1}\sum_{i=1}^{n} \mathsf{I}(X_{ji} \leq x),\\
\widetilde{F}_j(x) &= \frac{1}{n+1}\sum_{i=1}^{n} K\left(\frac{x - X_{ji}}{h}\right)
\end{aligned}
$$

for $j = 1, \dots, k$, where $\varkappa : \mathbb{R} \to \mathbb{R}$, $\int \varkappa = 1$, $K(x) = \int_{-\infty}^{x} \varkappa(t)dt$ and $h > 0$ is the bandwidth.

$$
\check{F}_j(x) = \{\widehat{F}_j(x), \widetilde{F}_j(x), F_j(x;\widehat{\alpha}_j)\}
$$

Estimation Issues

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\theta}_1^\top}, \dots, \frac{\partial \mathcal{L}_p}{\partial \boldsymbol{\theta}_p^\top}\right)^\top = \mathbf{0},$$

$$\begin{aligned} \text{where} \quad \mathcal{L}_j \quad &= \sum_{i=1}^{n} l_j(\mathbf{X}_i) \\ l_j(\mathbf{X}_i) \quad &= \log\left[c(\{\phi_\ell, \boldsymbol{\theta}_\ell\}_{\ell=1,\dots,j};\, s_j)\left(\{\check{F}_m(x_{mi})\}_{m\in s_j}\right)\right] \\ &\quad \text{for } j = 1, \dots, p. \end{aligned}$$

Estimation Issues

Nonparametric Estimation

$$\widehat{C}(u_1, \ldots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathrm{I}\{\breve{F}_j(X_{ji}) \leq u_j\}$$

$$\widetilde{C}(u_1, \ldots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} K_j \left\{ \frac{u_j - \breve{F}_j(X_{ji})}{h_j} \right\}$$

where $\breve{F}_j(x) = \{\widehat{F}_j(x),\ \widetilde{F}_j(x),\ F_j(x, \widehat{\alpha}),\ F_j(x)\}$

K-distribution

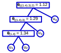
Let $V = C\{F_1(X_1), \dots, F_d(X_d)\}$ and let $K(t)$ denote the distribution function (K-distribution) of the random variable V.

We consider a HAC of the form $C_1\{u_1, C_2(u_2, \dots, u_d)\}$. Let $U_i \sim U[0,1]$ and let $V_2 = C_2(U_2, \dots, U_d) \sim K_2$.

Theorem

Let $U_1 \sim U[0,1]$, $V_2 \sim K_2$ and let $P(U_1 \le x, V_2 \le y) = C_1\{x, K_2(y)\}$ with $C_1(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0,1]$. Under certain regularity conditions the distribution function K_1 of the random variable $V_1 = C_1(U_1, V_2)$ is given by

$$K_1(t) = t - \int_0^{\phi^{-1}(t)} \phi'\left\{\phi^{-1}(t) + \phi^{-1} \circ K_2 \circ \phi(u) - u\right\} du \quad \text{for} \quad t \in [0,1].$$



Gumbel copula

$$
\begin{aligned}
\phi_\theta(t) &= \exp(-t^{1/\theta}), \\
\phi_\theta^{-1}(t) &= \{-\log(t)\}^{\theta}, \\
\phi'_\theta(t) &= -\frac{1}{\theta}\exp(-t^{1/\theta})t^{-1+1/\theta}.
\end{aligned}
$$

Following Genest and Rivest (1993), K for the simple 2-dim Archimedean copula with generator ϕ is given by $K(t) = t - \phi^{-1}(t)\phi'\{\phi^{-1}(t)\}$. Thus

$$
K_2(t,\theta) = t - \frac{t}{\theta}\log(t)
$$

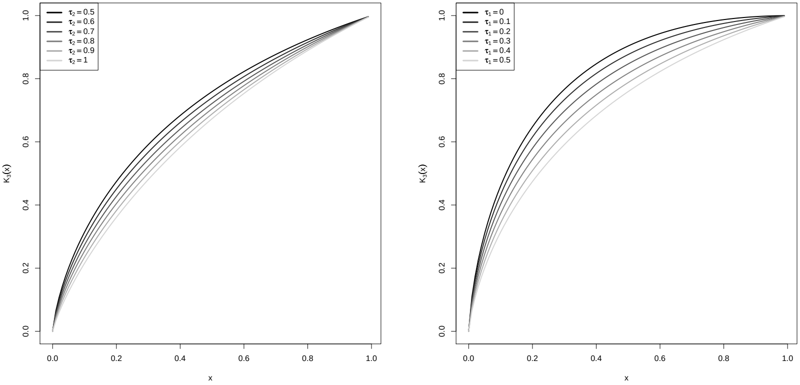


Figure 1: K distribution for three-dimensional HAC with Gumbel generators

Goodness-of-Fit Tests

$$H_0: \quad C \in \mathcal{C}_0, \quad \text{against} \quad H_1: \quad C \notin \mathcal{C}_0,$$

where $\mathcal{C}_0 = \{C_\theta : \theta \in \Theta\}$ is a known parametric family of copulae.

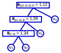
$$S = n \int_{[0,1]^d} \{\widehat{C}(u_1, \ldots, u_d) - C(u_1, \ldots, u_d, \widehat{\theta})\}^2 \, d\widehat{C}(u_1, \ldots, u_d),$$

$$T = \sup_{u_1, \ldots, u_d \in [0,1]} \sqrt{n} |\widehat{C}(u_1, \ldots, u_d) - C(u_1, \ldots, u_d, \widehat{\theta})|,$$

$$S_K = n \int_0^1 \{\widehat{K}(v) - K(v, \theta)\}^2 \, dv,$$

$$T_K = \sup_{v \in [0,1]} |\widehat{K}(v) - K(v, \theta)|.$$

where $\widehat{K}(v) = \frac{1}{n} \sum_{i=1}^n \mathrm{I}\{V_i \leq v\}$.



Simulation Study

1. F : two methods of estimation of margins (parametric and nonparametric);
2. C_0 : hypothesised copula models under H_0 (three models);
3. C : copula model from which the data were generated (three models with 3, 3 and 15 levels of dependence respectively);
4. n : size of each sample drawn from C (two possibilities, $n = 50$ and $n = 150$).

$\rightsquigarrow 2 \times 3 \times (3 + 3 + 15) \times 2 =$ 252 models with 100 repetitions

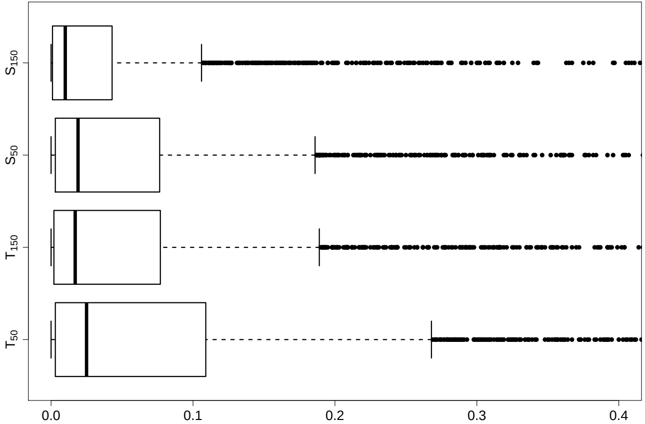


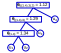
Figure 2: Levels of goodness-of-fit tests for different sample size, for parametric margins.

		AC							
		$n = 50$				$n = 150$			
θ		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\theta(0.25)$	HAC	0.88	0.51	0.83	0.38	0.93	0.36	0.90	0.35
	AC	0.88	0.51	0.89	0.50	0.95	0.32	0.90	0.34
	Gauss	0.71	0.29	0.56	0.22	0.69	0.11	0.43	0.08
$\theta(0.5)$	HAC	0.90	0.38	0.94	0.30	0.87	0.35	0.88	0.27
	AC	0.96	0.55	0.95	0.45	0.90	0.45	0.92	0.35
	Gauss	0.76	0.30	0.65	0.19	0.47	0.13	0.31	0.02
$\theta(0.75)$	HAC	0.93	0.29	0.93	0.15	0.89	0.27	0.89	0.10
	AC	0.93	0.29	0.93	0.22	0.90	0.25	0.91	0.13
	Gauss	0.77	0.19	0.65	0.10	0.57	0.11	0.24	0.05

Table 1: Non-rejection rate of the different models, where the sample is drawn from the simple AC

		HAC							
		$n = 50$				$n = 150$			
θ		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\theta(0.25, 0.5)$	HAC	0.88	0.29	0.90	0.24	0.96	0.31	0.92	0.26
	AC	0.91	0.26	0.93	0.36	0.54	0.13	0.53	0.07
	Gauss	0.82	0.20	0.69	0.19	0.57	0.14	0.37	0.04
$\theta(0.25, 0.75)$	HAC	0.93	0.21	0.92	0.13	0.88	0.18	0.88	0.09
	AC	0.46	0.14	0.54	0.07	0.00	0.00	0.00	0.00
	Gauss	0.84	0.19	0.71	0.13	0.52	0.10	0.42	0.01
$\theta(0.5, 0.75)$	HAC	0.86	0.31	0.87	0.18	0.91	0.20	0.94	0.08
	AC	0.89	0.36	0.92	0.28	0.44	0.04	0.47	0.02
	Gauss	0.70	0.19	0.55	0.12	0.50	0.11	0.30	0.05

Table 2: Non-rejection rate of the different models, where the sample is drawn from the HAC



Gauss									
		$n = 50$				$n = 150$			
Σ		T		S		T		S	
		emp.	par.	emp.	par.	emp.	par.	emp.	par.
$\Sigma(0.25, 0.25, 0.75)$	HAC	0.89	0.20	0.93	0.11	0.78	0.08	0.81	0.02
	AC	0.43	0.13	0.47	0.09	0.00	0.00	0.00	0.00
	Gauss	0.88	0.22	0.89	0.12	0.87	0.11	0.86	0.03
$\Sigma(0.25, 0.75, 0.25)$	HAC	0.92	0.20	0.91	0.14	0.76	0.07	0.69	0.04
	AC	0.39	0.12	0.39	0.04	0.00	0.00	0.00	0.00
	Gauss	0.90	0.18	0.87	0.13	0.92	0.12	0.94	0.10
$\Sigma(0.75, 0.25, 0.25)$	HAC	0.89	0.30	0.93	0.16	0.78	0.10	0.75	0.04
	AC	0.51	0.16	0.46	0.07	0.00	0.00	0.00	0.00
	Gauss	0.91	0.28	0.90	0.17	0.88	0.13	0.86	0.06

Table 3: Non-rejection rate of the different models, where the sample is drawn from the Gaussian copula

Data and Copula

- daily returns of Bank of America, Citigroup, Santander
- timespan = [29.09.2000 - 16.02.2001] ($n = 100$)
- ARMA(1,1)-GARCH(1,1)-residuals are conditionally distributed with estimated copula

$$
\begin{aligned}
R_{tj} &= \mu_j + \gamma_j R_{t-1,j} + \zeta_j \sigma_{t-1,j}\varepsilon_{t-1,j} + \sigma_{tj}\varepsilon_{tj}, \\
\sigma^2_{tj} &= \omega_j + \alpha_j \sigma^2_{t-1,j} + \beta_j \sigma^2_{t-1,j}\varepsilon^2_{t-1,j} \\
\varepsilon &\sim C\{F_1(x_1), \dots, F_d(x_d); \theta_t\}
\end{aligned}
$$

where $F_1, \dots, F_d$ are marginal distributions and θ_t are the copula parameters and $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $\alpha_j + \beta_j < 1$, $|\zeta| < 1$.

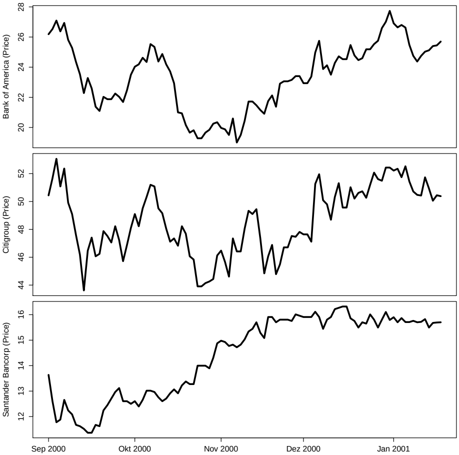


Figure 3: Stock prices for Bank of America, Citigroup and Santander (from top to bottom).

	$\widehat{\mu}_j$	$\widehat{\gamma}_j$	$\widehat{\zeta}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\beta}_j$
Bank of America	1.87e-3	0.22	-0.23	3.46e-4	0.55	0.17
(0.57, 0.83)	(2.59e-3)	(0.64)	(0.65)	(1.37e-04)	(0.28)	(0.16)
Citigroup	0.11e-3	0.31	-0.46	2.67e-4	0.09	0.47
(0.57, 0.79)	(1.48e-3)	(0.29)	(0.29)	(5.53e-04)	(0.17)	(1.01)
Santander	1.35e-3	0.43	-0.56	4.51e-10	0.01	0.98
(0.91, 0.78)	(0.91e-3)	(0.15)	(0.17)	(1.38e-05)	(0.02)	(0.05)

Table 4: Fitting of univariate ARMA(1,1)-GARCH(1,1) to asset returns. The standard deviation of the parameters, which are quiet big because of the small sample size, are given in parentheses. Each second row provides the p-values of the Box-Ljung test (BL) for autocorrelations and Kolmogorov-Smirnov test (KS) for testing of normality of the residuals.

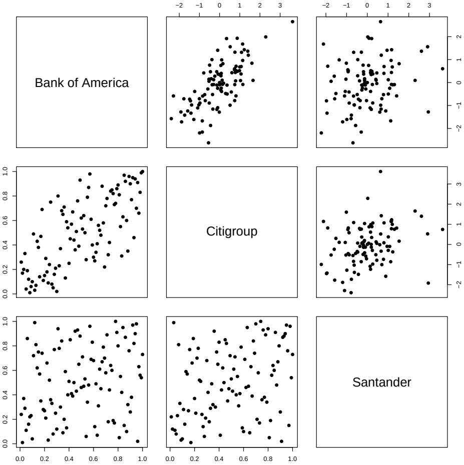


Figure 4: Scatterplots from ARMA-GARCH residuals (upper triangular) and from residuals mapped on unit square by the cdf (lower triangular).

	T_{100}	S_{100}	estimates
HAC	0.3191	0.1237	$C\{C(u_1, u_2; 1.996), u_3; 1.256\}$
AC	0.0012	0.0002	$C(u_1, u_2, u_3; 1.276)$
Gauss	0.0160	0.0078	$C_N\{u_1, u_2, u_3; \Sigma(0.697, 0.215, 0.312)\}$

Table 5: p-values of both GoFs and estimates of the models under different H_0 hypotheses.

The value V_t of the portfolio $w = \{w_1, \ldots, w_d\},\ w_i \in \mathbb{Z}$ is given by

$$V_t = \sum_{j=1}^{d} w_j X_{tj} \tag{2}$$

and the profit and loss (P&L) function of the portfolio

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{j=1}^{d} w_j X_{tj} \left\{ \exp(R_{t+1,j}) - 1 \right\} \tag{3}$$

The distribution function of L is given by

$$F_L(x) = P(L \leq x). \tag{4}$$

As usual the Value-at-Risk at level α from a portfolio w:

$$\mathrm{VaR}(\alpha) = F_L^{-1}(\alpha). \tag{5}$$

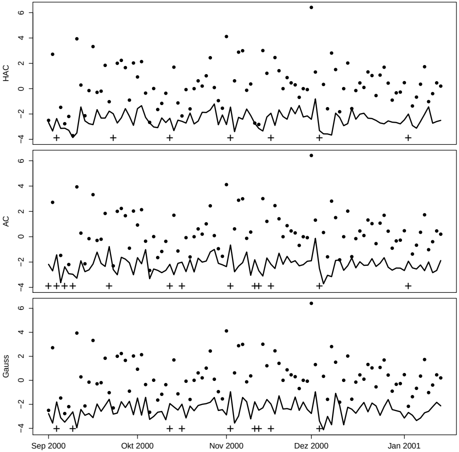


Figure 5: Profit and loss function and VaR based on different models.

α	$\widehat{\alpha}_{HAC}$	$\widehat{\alpha}_{AC}$	$\widehat{\alpha}_{Gauss}$
0.10	0.091	0.122	0.081
0.05	0.040	0.061	0.031
0.01	0.000	0.010	0.000

Table 6: Backtesting for the estimation of VaR under different alternatives.

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