Local Volatility Dynamics of LETF options

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Implied & Local Volatility

- Implied Volatility (IV): $\widehat{\sigma} : (t, K, T) \to \widehat{\sigma}_t(K, T)$: the BS option price implied measure of volatility
- local volatility (LV): under risk-neutral measure Q assume

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t, \cdot)dW_t^{\mathrm{Q}}, \tag{1}$$

then the local volatility is given by

$$\sigma_{K,T}(S_t, t) \stackrel{\text{def}}{=} \sqrt{\sigma^2_{K,T}(S_t, t)} = \sqrt{\mathrm{E}^{\mathrm{Q}}\{\sigma^2(S_T, T, \cdot)|S_T = K, \mathcal{F}_t\}} \tag{2}$$

with $(\Omega, \mathcal{F}, P)$ probability space, $\{\mathcal{F}_t\}_{0 \le t \le T}$ filtration

Why study local volatility?

- Advantages
  - easier calibration
  - LV allows pricing using entire volatility smile
  - therefore LV the correct input parameter for pricing exotic options, Derman, Kani & Zou (1996)
- Disadvantages
  - static nature in many cases
  - forward volatility flattening-out

(L)ETFs

- Exchange-traded funds (ETFs): tracking returns on financial quantities and yielding the identical daily return,
  e.g., SPDR S&P 500 ETF (SPY) tracks the S&P 500.
- Leveraged exchange-traded funds (LETFs): promising a fixed leverage ratio β w.r.t. a given underlying asset or index,
  e.g.,

LETF	β
ProShares Ultra S&P500 (SSO)	2
ProShares UltraPro S&P 500 (UPRO)	3
ProShares UltraShort S&P500 (SDS)	-2
ProShares UltraPro Short S&P 500 (SPXU)	-3

Table 1: LETFs with different β ▸ Illustration

Go with market

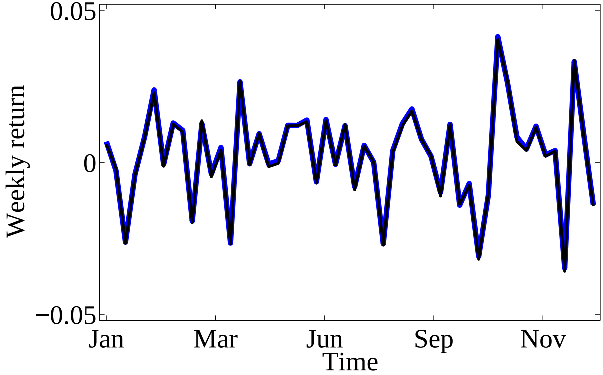


Figure 1: Weekly returns of ETF (SPY) and stock market (S&P 500) (20140101-20141230)

Leverage up/down

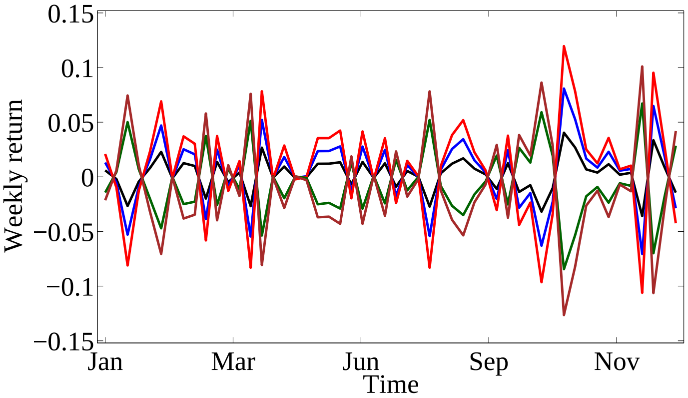


Figure 2: Weekly returns of LETFs (SSO, UPRO, SDS, SPXU) and stock market (S&P 500) (20140101-20141230)

Implied volatility paradoxon

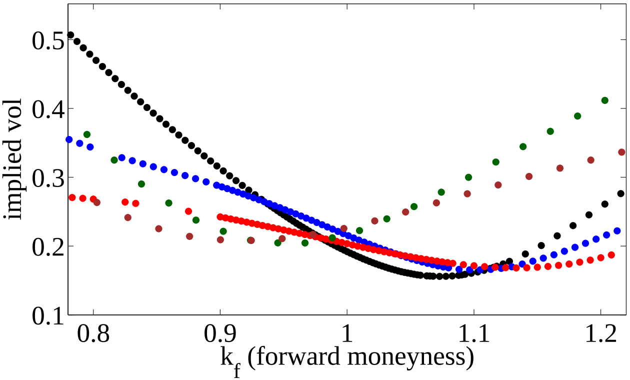


Figure 3: Implied volatility of (L)ETF options (SPY, SSO, UPRO, SDS, SPXU) with 21 days to maturity

Objectives

- introduce a dynamic model for IVS/LVS
- model IV and LV for LETF options with ETF option data
- identify LETF option price discrepancies using moneyness scaling

Outline

1. Motivation ✓
2. Moneyness scaling
3. DSFM Model
4. Estimation Results
5. Outlook

LETFs and the Black-Scholes model

- asset price dynamics:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathsf{Q}} \tag{3}$$

with interest rate r and volatility σ; W_t^{Q} standard Brownian motion under the risk-neutral measure P^{Q}

- (L)ETF dynamics:

$$\begin{aligned} \frac{dL_t}{L_t} &= \beta \left(\frac{dS_t}{S_t} \right) - \{(\beta - 1)r + c\}dt \\ &= (r - c)dt + \beta\sigma dW_t^{\mathsf{Q}} \end{aligned} \tag{4}$$

$0 \leq c \ll r$ (L)ETF expense ratio

Dupire formula

LV estimated using the Dupire formula, Fengler (2005):

$$\widehat{\sigma}^2_{K,T}(S_t, t) = \frac{\frac{\widehat{\sigma}}{\tau} + 2\frac{\partial \widehat{\sigma}}{\partial T} + 2K(r-\delta)\frac{\partial \widehat{\sigma}}{\partial K}}{K^2 \left\{ \frac{1}{K^2 \widehat{\sigma}\tau} + 2\frac{d_1}{K\widehat{\sigma}\sqrt{\tau}}\frac{\partial \widehat{\sigma}}{\partial K} + \frac{d_1 d_2}{\widehat{\sigma}}\left(\frac{\partial \widehat{\sigma}}{\partial K}\right)^2 + \frac{\partial^2 \widehat{\sigma}}{\partial K^2} \right\}}$$

where $d_1 \stackrel{\text{def}}{=} \frac{\log(S_t/K) + (r - \delta + \frac{1}{2}\widehat{\sigma}^2)\tau}{\widehat{\sigma}\sqrt{\tau}}$, $d_2 \stackrel{\text{def}}{=} d_1 - \widehat{\sigma}\sqrt{\tau}$, δ dividend rate, $\tau \stackrel{\text{def}}{=} T - t$, $\widehat{\sigma} = \widehat{\sigma}_t(K, T)$

LV for LETF options

Dupire formula:

$$(\beta\sigma_{K,T})^2 = 2\frac{\frac{\partial C_{BS}^{\beta}}{\partial T} + cC_{BS}^{\beta} + (r-c)K\frac{\partial C_{BS}^{\beta}}{\partial K}}{K^2\frac{\partial^2 C_{BS}^{\beta}}{\partial K^2}},$$

hence

$$\sigma_{K,T}^2 = 2\frac{\frac{\partial C_{BS}^{\beta}}{\partial T} + cC_{BS}^{\beta} + (r-c)K\frac{\partial C_{BS}^{\beta}}{\partial K}}{\beta^2 K^2\frac{\partial^2 C_{BS}^{\beta}}{\partial K^2}} \qquad (5)$$

LV for LETF options

- assume:

$$C_{BS}^{\beta}(t, L; K, T) \stackrel{\text{def}}{=} C^{BS}(t, L; K, T, r, c, |\beta|\sigma) \tag{6}$$
$$= C^{BS}\{t, L; K, T, r, c, |\beta|\widehat{\sigma}(K, T)\}$$

- estimate LV using an estimator of IV $\widehat{\sigma}$:

$$\widehat{\sigma}^2_{K,T} = \frac{\frac{\widehat{\sigma}}{2\tau} + \frac{\partial\widehat{\sigma}}{\partial T} + (r-c)K\frac{\partial\widehat{\sigma}}{\partial K}}{\frac{1}{2}K^2\left\{\frac{1}{K^2\widehat{\sigma}\tau} + \frac{2|\beta|d_1}{K\widehat{\sigma}\sqrt{\tau}}\frac{\partial\widehat{\sigma}}{\partial K} + \frac{|\beta|^2 d_1 d_2}{\widehat{\sigma}}\left(\frac{\partial\widehat{\sigma}}{\partial K}\right)^2 + |\beta|^2\frac{\partial^2\widehat{\sigma}}{\partial K^2}\right\}} \tag{7}$$

where $d_1 = \frac{\log(L_t/K) + (r - c + \frac{1}{2}|\beta|^2\widehat{\sigma}^2)\tau}{|\beta|\widehat{\sigma}\sqrt{\tau}}$ and $d_2 = d_1 - |\beta|\widehat{\sigma}\sqrt{\tau}$.

▸ Details

Moneyness scaling

- link LV to IV in terms of the forward moneyness measure $\kappa_f \stackrel{\text{def}}{=} K/\{e^{(r-c)\tau}L_t\}$ and time to maturity τ:

$$\widehat{\sigma}^2_{\kappa_f,\tau} = \frac{\widehat{\sigma}^2 + 2\tau\widehat{\sigma}\frac{\partial\widehat{\sigma}}{\partial\tau}}{1 + 2|\beta|\kappa_f\sqrt{\tau}\widetilde{d}_1\frac{\partial\widehat{\sigma}}{\partial\kappa_f} + |\beta|^2(\kappa_f)^2\tau\left\{\widetilde{d}_1\widetilde{d}_2(\frac{\partial\widehat{\sigma}}{\partial\kappa_f})^2 + \widehat{\sigma}\frac{\partial^2\widehat{\sigma}}{\partial\kappa_f^2}\right\}} \tag{8}$$

where $\widetilde{d}_1 = \frac{-\log(\kappa_f)+\frac{1}{2}|\beta|^2\widehat{\sigma}^2\tau}{\beta\widehat{\sigma}\sqrt{\tau}}$ and $\widetilde{d}_2 = \widetilde{d}_1 - |\beta|\widehat{\sigma}\sqrt{\tau}$.

- link moneyness axis for LETF options with leverage β_1, β_2

Details

$$\kappa_f{}^{(\beta_1)} = \exp\left\{-\frac{\beta_1}{2}(\beta_1 - \beta_2)\overline{\sigma}^2\tau\right\}\left(\kappa_f{}^{(\beta_2)}\right)^{\frac{\beta_1}{\beta_2}}, \tag{9}$$

where $\overline{\sigma}$ is the average IV across all strikes

Simulation example

- level-dependent local volatility dynamics:

$$\frac{dL_t}{L_t} = (r - c)dt + \beta\sigma(t, S_t)dW_t^{\mathsf{Q}},$$

where $\sigma(t, S) = \kappa S^{-\gamma}$, $\kappa = 0.2$, $\gamma = 3.5$

- price LETF option by the Monte-Carlo
- model IV using LPS Details
- calculate LV by (8) and do moneyness scaling by (9)
- plot IV and LV against κ_f before and after moneyness scaling Show

Real data example

- data: IV, strikes, maturities for SPY, SSO, UPRO, SDS and SPXU options
- data source: Datastream
- IV and LV against forward moneyness for real data ▸ Show

Challenges for IV/LV estimation

- how to model the IV/LV surface?
- degenerated data design: IVS observations only for a small number of maturities
- observation grid does not cover desired estimation grid
  - the contracts are not traded for a particular strike
  - institutional arrangements at the futures' exchanges

Implied volatility in time

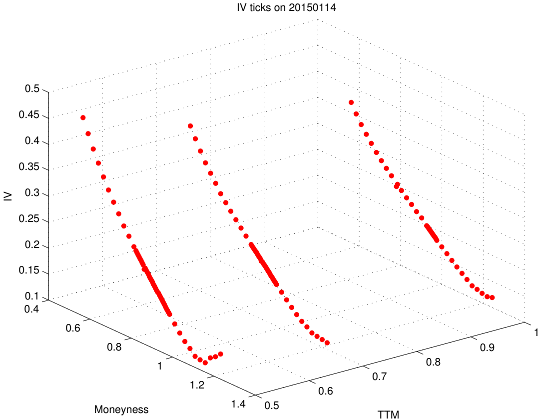


Figure 4: SPY ETF option IV ticks of 20150114-20150408

IV data design

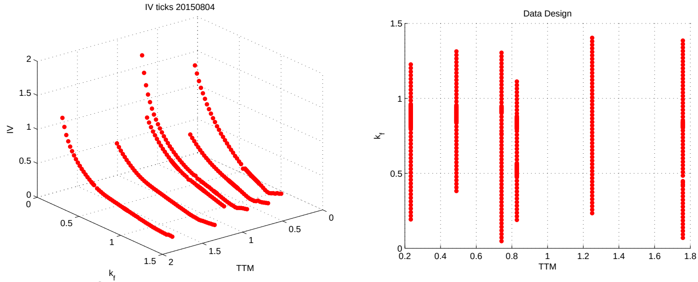


Figure 5: *Left panel*: SPY call IV observed on 20150408; *Right panel*: data design on 20150408

The dynamic semiparametric factor model for IVS

- define $\mathcal{J} \stackrel{\text{def}}{=} [\kappa_{f\,min}, \kappa_{f\,max}] \times [\tau_{min}, \tau_{max}]$, $Y_{i,j}$ implied volatility, $i = 1, \ldots, I$ time index, $j = 1, \ldots, J_i$ option intraday numbering on day i, $X_{i,j} \stackrel{\text{def}}{=} (\kappa_{f\,i,j}, \tau_{i,j})^\top$, $Y_{i,j} \stackrel{\text{def}}{=} \sigma_{i,j}$ implied volatility

- assume

$$Y_{i,j} = \mathcal{Z}_i^\top m(X_{i,j}) + \varepsilon_{i,j}, \tag{10}$$

  where $\mathcal{Z}_i = (1, Z_i^\top)$, $Z_i = (Z_{i,1}, \ldots, Z_{i,L})^\top$ unobservable L-dimensional process, $m = (m_0, \ldots, m_L)^\top$, real-valued functions m_l, $l = 1, \ldots, L+1$ defined on a subset of $\mathbb{R}^d$

- $X_{i,j}$, $\varepsilon_{i,j}$ are independent, $\varepsilon_{i,j} \sim (0, \sigma^2)$, $\sigma^2 < \infty$

DSFM for IVS

Approximate, Park et al. (2009):

$$\mathsf{E}(Y_i|X_i) = \mathcal{Z}_i^\top m(X_i) = \mathcal{Z}_i^\top \mathcal{A}\psi(X_i), \tag{11}$$

where

$$\psi(X_i) \stackrel{\text{def}}{=} \{\psi_1(X_i), \ldots, \psi_K(X_i)\}^\top \text{ space basis,}$$
$$\mathcal{A} : (L+1) \times K \text{ coefficient matrix}$$

Choose $\{\psi_k : 1 \leq k \leq K\}$ tensor B-spline basis ▸ Details, de Boor (2001)

Tensor B-spline basis

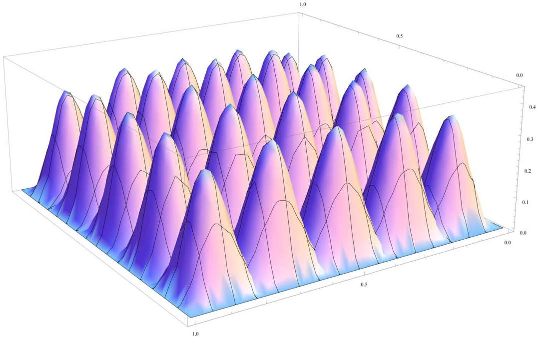


Figure 6: Tensor B-spline basis with 15×15 knots on $[0,1] \times [0,1]$, odd intervals

Estimation

Define the least-squares estimators $\widehat{Z}_i = (\widehat{Z}_{i,1}, \ldots, \widehat{Z}_{i,L})^\top$, $\widehat{\mathcal{A}} = (\widehat{\alpha}_{l,k})_{l=0,\ldots,L;k=1,\ldots,K}$

$$(\widehat{Z}_i, \widehat{\mathcal{A}}) = \arg\min_{Z_i, \mathcal{A}} S(\mathcal{A}, Z), \tag{12}$$

where

$$S(\mathcal{A}, Z) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \left\{ Y_{i,j} - (1, Z_i^\top)\mathcal{A}\psi(X_{i,j}) \right\}^2 \tag{13}$$

Once $\widehat{\mathcal{A}}$ obtained, m can be estimated as $\hat{m} = \widehat{\mathcal{A}}\psi$

Identification

- the problem (12) can be solved via numeric algorithm ▸ Details
- under certain conditions ▸ Details, geometric convergence to a solution
- (12) has no unique solution: orthonormalize $\widehat{Z}_i$, $\hat{m}$ for better interpretation, see Fengler et al. (2007)

Data overview

	Min.	Max.	Mean	Stdd.	Skewn.	Kurt.
TTM	0.20	0.70	0.53	0.14	−0.43	2.04
Moneyness	0.65	1.25	0.92	0.13	0.54	3.10
IV	0.09	0.47	0.20	0.05	0.43	3.63

Table 2: Summary statistics on the SPY ETF option from 20141006 to 20150408 (in total $\sum_i J_i = 9452$ datapoints). Source: Datastream

Estimation

- data transformed with marginal empirical distribution functions
- 10 knots in moneyness and 7 knots in maturity direction: $K = 10 \times 7 = 70$
- starting values for Z_i generated from a stable VAR process
  Details
- starting values for $\mathcal{A}$ randomly generated from $U(0, 1)$
- convergence tolerance for the Newton algorithm: $1e$-06

Model order selection

Select model order by explained variance $EV(L)$

$$EV(L) \stackrel{\text{def}}{=} 1 - \frac{\sum_{i=1}^{I}\sum_{j=1}^{J_i}\left\{Y_{i,j} - \sum_{l=0}^{L}\widehat{Z}_{i,l}\hat{m}_l(X_{i,j})\right\}^2}{\sum_{i=1}^{I}\sum_{j=1}^{J_i}(Y_{i,j} - \overline{Y})^2} \qquad (14)$$

No. factors	$EV(L)$
3	0.912
4	0.916
5	0.924
6	0.925

Estimate $L = 3$ basis functions

Dynamics of $\widehat{Z}_i$

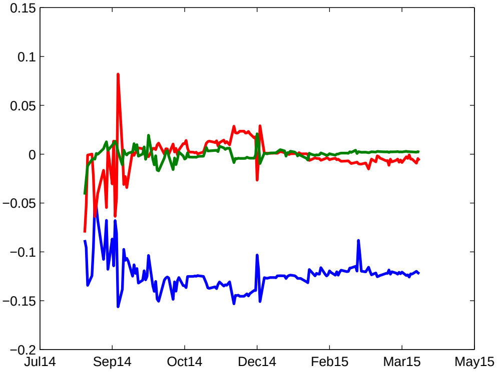


Figure 7: Time dynamics of $\widehat{Z}_{i,1}$, $\widehat{Z}_{i,2}$, $\widehat{Z}_{i,3}$

VAR modelling of $\widehat{Z}_i$

- Hannan-Quinn criterion selects the VAR(3) model; Schwarz the VAR(1)
- roots lay inside the unit circle
- Portmanteau and Breusch-Godfrey LM test results with 20 lags reject residual autocorrelation

VAR parameters

	$\widehat{Z}_{1,t-1}$	$\widehat{Z}_{2,t-1}$	$\widehat{Z}_{3,t-1}$	$\widehat{Z}_{1,t-2}$	$\widehat{Z}_{2,t-2}$
$\widehat{Z}_{1,t}$	0.90	0.21	−0.66	−0.03	0.10
$\widehat{Z}_{2,t}$	−0.48	0.02	0.44	−0.11	−0.21
$\widehat{Z}_{3,t}$	−0.15	−0.15	0.51	0.13	0.10

	$\widehat{Z}_{3,t-2}$	$\widehat{Z}_{1,t-3}$	$\widehat{Z}_{2,t-3}$	$\widehat{Z}_{3,t-3}$	c
$\widehat{Z}_{1,t}$	0.04	0.20	−0.06	−0.18	0.01
$\widehat{Z}_{2,t}$	0.06	−0.12	0.14	0.23	−0.09
$\widehat{Z}_{3,t}$	−0.22	0.04	−0.01	0.19	0.00

Table 3: VAR(3) estimated parameters for Z_i

Estimated factor functions $\hat{m}$

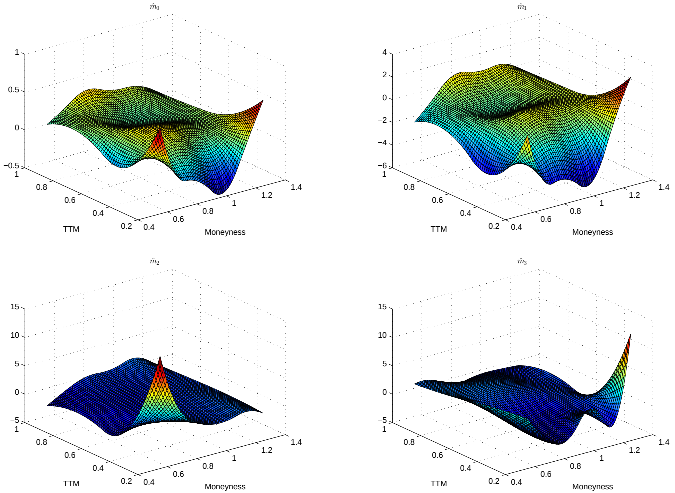
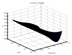


Figure 8: Factor functions $\hat{m}_0$, $\hat{m}_1$, $\hat{m}_2$, $\hat{m}_3$



DSFM IV Dynamics

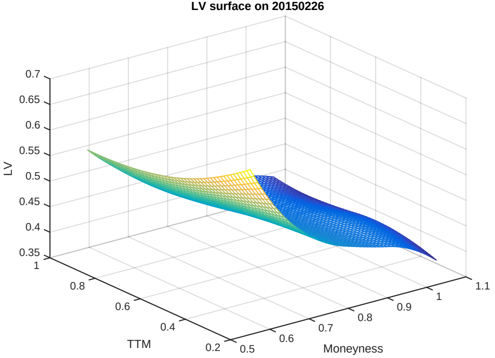


Figure 9: SPY ETF option IV dynamics in 20150226-20150325

Bias comparison

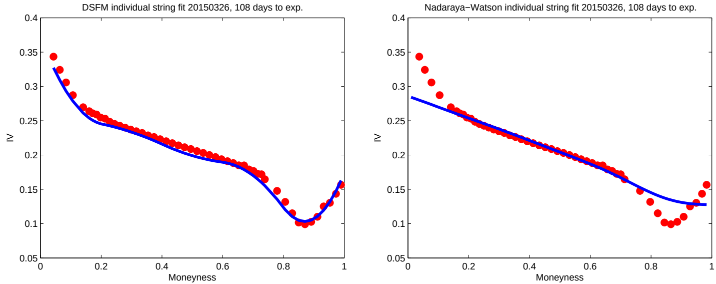


Figure 10: Bias comparison of the DSFM (*left panel*) and the Nadaraya-Watson estimator with $\hat{h} = (0.13, 0.12)^\top$, $\hat{h}$ by Scott's rule, for the 108 days to expiry data (red dots) on 20150326

Backing out LVS

- to avoid presence of arbitrage, arbitrage-free smoothing of the IVS can be done, see Fengler (2009)
- finite-difference methods ▸ Details to obtain $\frac{\partial \widehat{\sigma}}{\partial \tau}$, $\frac{\partial \widehat{\sigma}}{\partial k_f}$, $\frac{\partial^2 \widehat{\sigma}}{\partial k_f^2}$
- use the B-spline representation of $\widehat{\sigma}$ ▸ Details:

$$\frac{\partial^{q+r}\widehat{\sigma}_i}{\partial \kappa_f^q \partial \tau^r} = \frac{\partial^{q+r} m_0(\kappa_f, \tau)}{\partial \kappa_f^q \partial \tau^r} + \sum_{l=1}^{L} Z_{i,l} \frac{\partial^{q+r} m_l(\kappa_f, \tau)}{\partial \kappa_f^q \partial \tau^r} \tag{15}$$

- then $\frac{\partial \widehat{\sigma}}{\partial \tau}$, $\frac{\partial \widehat{\sigma}}{\partial k_f}$, $\frac{\partial^2 \widehat{\sigma}}{\partial k_f^2}$ follow as special cases of (15)

DSFM LVS Dynamics

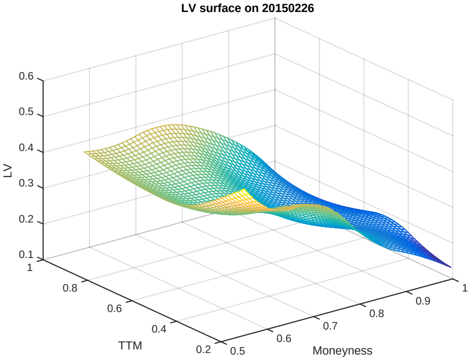
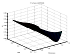


Figure 11: SPY ETF option LV dynamics in 20150226-20150325



LETF IV, LV dynamics from ETF data

- assume $\beta_1 = +2$: long leveraged position
- apply moneyness scaling (9) to the original SPY ETF κ_f data with $\beta_2 = 1$, obtain $\kappa_f^{(\beta_1)}$
- do DSFM estimation (12) with $\kappa_f = \kappa_f^{(\beta_1)}$
- plot the IVs against the original τ, $\kappa_f^{(\beta_2)}$
- estimate the LVs with the Dupire formula (8)

DSFM LETF IV dynamics

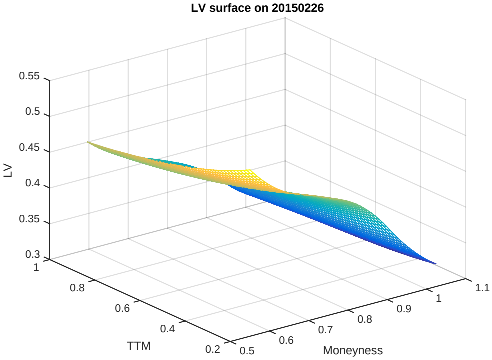


Figure 12: $\beta = +2$ LETF option IV dynamics in 20150226-20150325

DSFM LETF LV dynamics

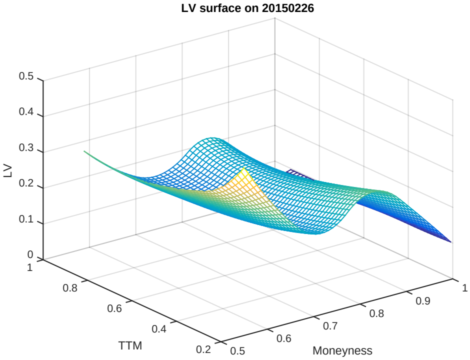


Figure 13: $\beta = +2$ LETF option IV dynamics in 20150226-20150325

Conclusions and outlook

- the DSFM approach allows to model dynamic local volatility surfaces
- stable structure of the factor loadings allows IVS/LVS prediction
- construction of confidence bands for the mismatch of LV of LETF options
- identification of LETF option price discrepancies across different strikes and maturities using moneyness scaling
- expansion to a stochastic local volatility model

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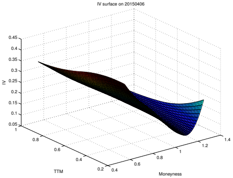

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S&P 500 and (L)ETFs Return

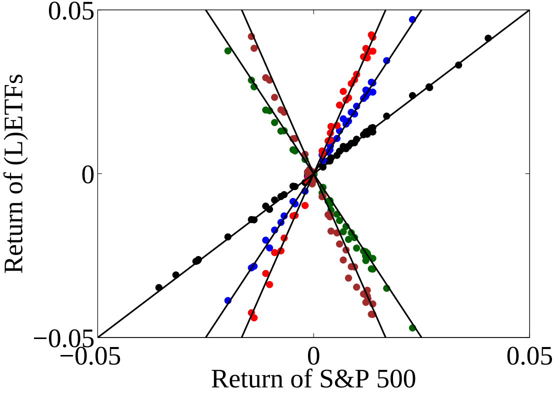


Figure 14: Weekly return relationship of S&P 500 and (L)ETFs (SPY, SSO, UPRO, SDS, SPXU) (20140101-20141230) Back

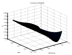
Local Polynomial Smoothing (LPS) for IVS

$$\min_{\alpha(x_1,x_2)\in\mathbb{R}^5} \sum_{i=1}^{n} \{y_i - \alpha_0 - \alpha_1(x_{1i} - x_1) - \alpha_2(x_{2i} - x_2) - \alpha_3(x_{1i} - x_1)^2 - \alpha_4(x_{2i} - x_2)^2 - \alpha_5(x_{1i} - x_1)(x_{2i} - x_2)\}^2 \frac{K(\frac{x_1 - x_{1i}}{h_1})}{h_1} \frac{K(\frac{x_2 - x_{2i}}{h_2})}{h_2},$$

where $K(u) = \frac{3}{4}(1 - u^2)I(|u| < 1)$ Epanechnikov kernel;

$$\frac{\partial y}{\partial x_1}\,|_{(x_{1i},x_{2i})} = \alpha_1, \;\; \frac{\partial y}{\partial x_2}\,|_{(x_{1i},x_{2i})} = \alpha_2, \;\; \frac{\partial^2 y}{\partial {x_1}^2}\,|_{(x_{1i},x_{2i})} = 2\alpha_3$$

Return to "Simulation example"



IV and LV against forward moneyness

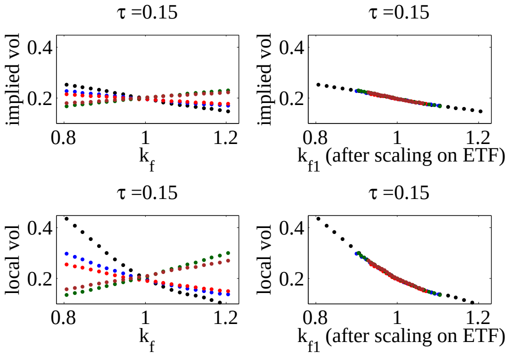


Figure 15: IV and LV against forward moneyness for (L)ETF options with $\beta = 1$, $\beta = 2$, $\beta = 3$, $\beta = -2$ and $\beta = -3$. Back

IV and LV against forward moneyness for real data

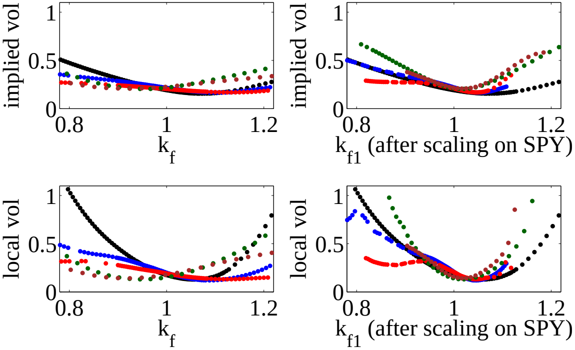


Figure 16: IV and LV against forward moneyness for S&P 500 (L)ETF options (SSO, UPRO, SDS, SPXU) with 21 days to maturity Back

BS formula and its derivatives for LETF options

Recall (6), the BS is

$$C_{BS}^{\beta} = e^{-c\tau} L_t \Phi(d_1) - e^{-r\tau} K \Phi(d_2) \tag{16}$$

where $\Phi(\cdot)$ is the cdf of $N(0,1)$.

$$\begin{aligned} \frac{\partial C_{BS}^{\beta}}{\partial T} &= \frac{\partial C^{BS}}{\partial T} + \frac{\partial C^{BS}}{\partial \sigma} |\beta| \frac{\partial \widehat{\sigma}}{\partial T} \\ &= \frac{e^{-c\tau} L_t |\beta| \widehat{\sigma} \Phi(d_1)}{2\sqrt{\tau}} - c e^{-c\tau} L_t \Phi(d_1) + r e^{-r\tau} K \Phi(d_2) \\ &\quad + \frac{\partial C^{BS}}{\partial \sigma} |\beta| \frac{\partial \widehat{\sigma}}{\partial T} \end{aligned} \tag{17}$$

BS formula and its derivatives for LETF option

$$
\begin{aligned}
\frac{\partial C^{\beta}_{BS}}{\partial K} &= \frac{\partial C^{BS}}{\partial K} + \frac{\partial C^{BS}}{\partial \sigma}|\beta|\frac{\partial \widehat{\sigma}}{\partial K} \\
&= -e^{-r\tau}\Phi(d_2) + \frac{\partial C^{BS}}{\partial \sigma}|\beta|\frac{\partial \widehat{\sigma}}{\partial K}
\end{aligned} \tag{18}
$$

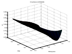
$$
\begin{aligned}
\frac{\partial^2 C^{\beta}_{BS}}{\partial K^2} &= \frac{\partial^2 C^{BS}}{\partial K^2} + 2\frac{\partial^2 C^{BS}}{\partial K \partial \widehat{\sigma}}|\beta|\frac{\partial \widehat{\sigma}}{\partial K} \\
&\quad + \frac{\partial^2 C^{BS}}{\partial \sigma^2}|\beta|^2\left(\frac{\partial \widehat{\sigma}}{\partial K}\right)^2 + \frac{\partial C^{BS}}{\partial \sigma}|\beta|\frac{\partial^2 \widehat{\sigma}}{\partial K^2} \\
&= \frac{\partial C^{BS}}{\partial \sigma}\left\{\frac{1}{|\beta|K^2\widehat{\sigma}\tau} + \frac{2d_1}{K\widehat{\sigma}\sqrt{\tau}}\frac{\partial \widehat{\sigma}}{\partial K} + \frac{|\beta|d_1 d_2}{\widehat{\sigma}}\left(\frac{\partial \widehat{\sigma}}{\partial K}\right)^2 + |\beta|\frac{\partial^2 \widehat{\sigma}}{\partial K^2}\right\}
\end{aligned} \tag{19}
$$

BS formula and its derivatives for LETF option

Substitute (18)-(19) into (5), obtain the Dupire formula in terms of IV and its derivatives:

$$\widehat{\sigma}^2_{K,T} = \frac{\frac{\widehat{\sigma}}{2\tau} + \frac{\partial\widehat{\sigma}}{\partial T} + (r-c)K\frac{\partial\widehat{\sigma}}{\partial K}}{\frac{1}{2}K^2\left\{\frac{1}{K^2\widehat{\sigma}\tau} + \frac{2|\beta|d_1}{K\widehat{\sigma}\sqrt{\tau}}\frac{\partial\widehat{\sigma}}{\partial K} + \frac{|\beta|^2 d_1 d_2}{\widehat{\sigma}}\left(\frac{\partial\widehat{\sigma}}{\partial K}\right)^2 + |\beta|^2\frac{\partial^2\widehat{\sigma}}{\partial K^2}\right\}}$$

Back



Moneyness scaling

Given the general solution of (4):

$$L_T = L_t \exp\left\{(r-c)(T-t) - \frac{\beta^2}{2}\int_t^T \sigma_s^2 ds + \beta \int_t^T \sigma_s dW_s^*\right\}, \tag{20}$$

write (20) for $L_T^{(\beta_1)}$, $L_T^{(\beta_2)}$, obtain

$$\frac{L_T^{(\beta_1)}}{e^{(r-c)\tau} L_t^{(\beta_1)}} = \exp\left(-\frac{\beta_1^2}{2}\int_0^\tau \sigma_s^2 ds + \beta_1 \int_0^\tau \sigma_s dW_s\right) \tag{21}$$

$$\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}} = \exp\left(-\frac{\beta_2^2}{2}\int_0^\tau \sigma_s^2 ds + \beta_2 \int_0^\tau \sigma_s dW_s\right) \tag{22}$$

where σ_s is the instantaneous volatility at time s.

Moneyness scaling

From (22) follows:

$$\int_t^T \sigma_s dW_s^* = \frac{\log\left(\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}}\right) + \frac{\beta_2^2}{2}\int_0^\tau \sigma_s^2 ds}{\beta_2} \tag{23}$$

Substitute (23) into (21) to eliminate the stochastic term $\int_t^T \sigma_s dW_s^*$, obtain:

$$\frac{L_T^{(\beta_1)}}{e^{(r-c)\tau} L_t^{(\beta_1)}} = \exp\left\{-\frac{\beta_1}{2}(\beta_1 - \beta_2)\int_0^\tau \sigma_s^2 \mathrm{d}s\right\}\left\{\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}}\right\}^{\frac{\beta_1}{\beta_2}} \tag{24}$$

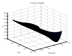
Moneyness scaling

Take logs and expectations conditioned on $K^{(\beta_1)} = L_T^{(\beta_1)}$ and $K^{(\beta_2)} = L_T^{(\beta_2)}$, obtain

$$\log({k_f}^{(\beta_1)}) = -\frac{\beta_1}{2}(\beta_1 - \beta_2)\mathrm{E}^* \left(\int_0^{\tau} \sigma_s^2 \mathrm{d}s \,\middle|\, K^{(\beta_1)} = L_T^{(\beta_1)}, K^{(\beta_2)} = L_T^{(\beta_2)} \right) + \frac{\beta_1}{\beta_2} \log({k_f}^{(\beta_2)})$$

Assuming constant σ and exponentiating, one obtains (9)

Return to "Moneyness scaling"



Tensor product B-splines

Define $U \stackrel{\text{def}}{=} \{\sum_i \alpha_i N_{i,h,s} : \alpha_i \in \mathbb{R}, i \in \mathbb{Z}\}$,
$V \stackrel{\text{def}}{=} \left\{\sum_j \beta_j N_{j,k,t} : \beta_j \in \mathbb{R}, j \in \mathbb{Z}\right\}$, then the tensor product B-spline is $b \stackrel{\text{def}}{=} \sum_{i,j} \gamma_{i,j} N_{i,j}$, $\gamma_{i,j} \in \mathbb{R}$, $w \in U \otimes V$, where

$$N_{i,j}(x,y) \stackrel{\text{def}}{=} N_{i,h,s}(x) N_{j,k,t}(y),$$

$$N_{i,k,t}(x) \stackrel{\text{def}}{=} \left(\frac{x - t_i}{t_{i+k} - t_i}\right) N_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) N_{i+1}^{k-1}(x),$$

with the starting point

$$N_{i,0,t}(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } t_i \le x < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

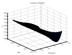
here $k \in \mathbb{N}$, t_i infinite set of knots Back to "DSFM for IVS"

Numerical differentiation

Use Taylor expansion for $\hat{\sigma}(\tau + h, \kappa_f)$, $\hat{\sigma}(\tau - h, \kappa_f)$, $\hat{\sigma}(\tau, \kappa_f + h)$, $\hat{\sigma}(\tau, \kappa_f - h)$, obtain the following approximations, h small:

- $$\frac{\partial \widehat{\sigma}}{\partial \tau} = \frac{\hat{\sigma}(\tau + h, \kappa_f) - \hat{\sigma}(\tau - h, \kappa_f)}{2h}$$
- $$\frac{\partial \widehat{\sigma}}{\partial \kappa_f} = \frac{\hat{\sigma}(\tau, \kappa_f + h) - \hat{\sigma}(\tau, \kappa_f - h)}{2h}$$
- $$\frac{\partial^2 \widehat{\sigma}}{\partial \kappa_f^2} = \frac{\hat{\sigma}(\tau, \kappa_f + h) - 2\hat{\sigma}(\tau, \kappa_f) + \hat{\sigma}(\tau, \kappa_f - h)}{h^2}$$

▸ Return to "Backing out LVS"



Tensor B-spline derivatives

- a B-spline surface $b(x,y)$ can be represented in Bézier form, Prautzsch et al. (2002)

$$b(x,y) = \sum_i \sum_j \beta_{ij} B_i^n(x) B_j^k(y), \tag{25}$$

where B_i^n are Bernstein polynomials ▸ Details

- the partial derivatives of (25) are given by

$$\frac{\partial^{q+r} b(x,y)}{\partial x^q \partial y^r} = \frac{n!k!}{n!k! - qr} \sum_i \sum_j \Delta^{01} \Delta^{q,r-1} \beta_{ij} B_i^{n-q}(x) B_j^{k-r}(y), \tag{26}$$

where the forward difference $\Delta^{qr}\beta_{ij} = \beta_{i+q,j+r} - \beta_{i,j}$

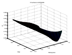
▸ Return to "Tensor B-spline derivatives"

Bernstein polynomials

Bernstein polynomials of degree n are given by

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i},$$

where $i = 0, \ldots, n$ Return to "Tensor B-spline derivatives"



Convergence conditions

Initial choice (α^0, Z^0) such that, Park et al. (2009):

A1 it holds that $\sum_{i=1}^{I} Z_i^0 = 0$; $\sum_{i=1}^{I} Z_i^0 {Z_i^0}^\top$ and the Hessian from (12) at (α^0, Z^0), $\mathcal{H}(\alpha^0, Z^0)$ are invertible

A2 there exists a version $(\widehat{\alpha}, \widehat{Z})$ with $\sum_{i=1}^{I} \widehat{Z}_i = 0$ such that $\sum_{i=1}^{I} \widehat{Z}_i {Z_i^0}^\top$ is invertible. Also, $\hat{\alpha}_l = (\hat{\alpha}_{l1}, \ldots, \hat{\alpha}_{lK})^\top$, $l = 0, \ldots, L$ are linearly independent

Return to "Identification"

Numeric algorithm I

The first-order conditions for (12):

$$\frac{\partial S(\mathcal{A}, Z)}{\partial \alpha} = 2\sum_{i=1}^{I}\left\{(\Psi_i\Psi_i^\top)\otimes(\mathcal{Z}_i\mathcal{Z}_i^\top)\right\}\alpha - 2\sum_{i=1}^{I}(\Psi_i Y_i)\otimes\mathcal{Z}_i, \tag{27}$$

$$\begin{aligned}\frac{\partial S(\mathcal{A}, Z)}{\partial Z} &= 2(\mathcal{Z}_1^\top\mathcal{A}\Psi_1\Psi_1^\top A^\top - Y_1^\top\Psi_1^\top A^\top, \ldots, \mathcal{Z}_I^\top\mathcal{A}\Psi_I\Psi_I^\top A^\top \\ &\quad - Y_I^\top\Psi_I^\top A^\top),\end{aligned} \tag{28}$$

where A is $\mathcal{A}$ without 1st row, $\Psi_i \stackrel{\text{def}}{=} \{\psi(X_{i,1}), \ldots, \psi(X_{i,J_i})\}$,
$\alpha \stackrel{\text{def}}{=} \text{vec}(\mathcal{A})$ Return to "Identification"

Numeric algorithm II

The second-order conditions for (12):

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial \alpha^2} = 2\sum_{i=1}^{I}\left\{(\Psi_i\Psi_i^\top)\otimes(\mathcal{Z}_i\mathcal{Z}_i^\top)\right\}, \tag{29}$$

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial Z^2} = \begin{pmatrix} A\Psi_1\Psi_1^\top A^\top & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A\Psi_I\Psi_I^\top A^\top \end{pmatrix}, \tag{30}$$

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial \alpha \partial Z} = 2\{F_1(\alpha, Z), \ldots, F_I(\alpha, Z)\}, \tag{31}$$

where

$F_i(\alpha, Z) \stackrel{\text{def}}{=} (\Psi_i\Psi_i^\top A^\top)\otimes \mathcal{Z}_i + (\Psi_i\Psi_i^\top \mathcal{A}^\top \mathcal{Z}_i)\otimes \mathcal{I} - (\Psi_i Y_i)\otimes \mathcal{I}$,

$\mathcal{I} = (0, I_L)$, I_L is $L \times L$ identity matrix Return to "Identification"

Numeric algorithm III

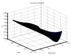
Collect the FOCs (27)-(28) and the SOCs (29)-(31) into the Newton iteration for (12):

$$x_{k+1} = x_k - \mathcal{H}^{-1}(x_k)\nabla(x_k), \tag{32}$$

where $x_k \stackrel{\text{def}}{=} \begin{pmatrix} \alpha^{(k)} \\ Z^{(k)} \end{pmatrix}$, $\mathcal{H}^{-1}(x_k) \stackrel{\text{def}}{=} \left.\begin{pmatrix} \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha^2} & \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha \partial Z} \\ \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha \partial Z}^\top & \frac{\partial^2 S(\mathcal{A},Z)}{\partial Z^2} \end{pmatrix}\right|_{x_k}$,

$\nabla(x_k) \stackrel{\text{def}}{=} \left.\begin{pmatrix} \frac{\partial S(\mathcal{A},Z)}{\partial \alpha} \\ \frac{\partial S(\mathcal{A},Z)}{\partial Z} \end{pmatrix}\right|_{x_k}$

▸ Return to "Identification"



Stable vector autoregressive process

VAR(p) process

$$y_t = c + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \quad t \in \mathbb{Z}, \tag{33}$$

where $y_t \in \mathbb{R}^k$ random vector, $A_i \in \mathbb{R}^{k \times k}$ fixed coefficient matrices, $c \in \mathbb{R}^k$ fixed vector of intercept terms, $u_t \in \mathbb{R}^k$ innovation process, $\mathsf{E}\, u_t = 0$, $\mathsf{E}\, u_t u_s^\top = 0$, $s \neq t$, $\Sigma_u \stackrel{\text{def}}{=} \mathsf{E}\, u_t u_t^\top$ is called *stable* if

$$\det(I_k - A_1 z - \cdots - A_p z^p) \neq 0 \quad \text{for} \quad |z| \leq 1,$$

i.e., the reverse characteristic polynomial of (33) has no roots inside and on the complex unit circle Return to "Estimation"