DYTEC - DYnamic Tail Event Curves and its applications

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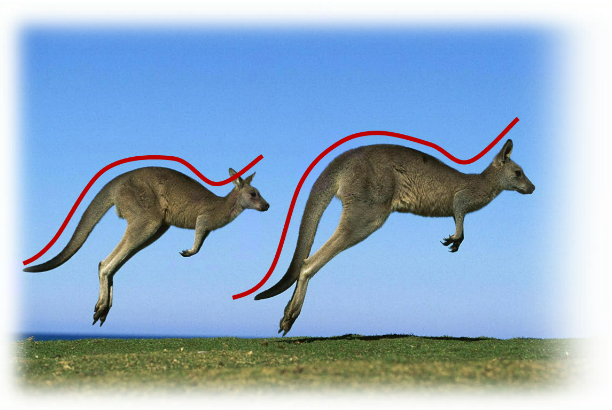
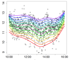
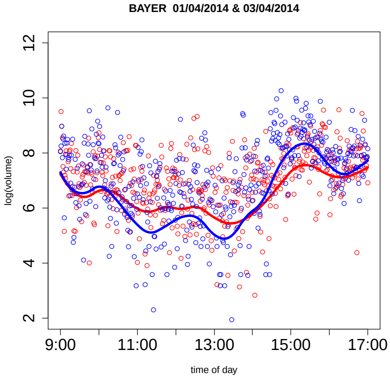
Dynamics of tail event curves





Intra-day trading volume

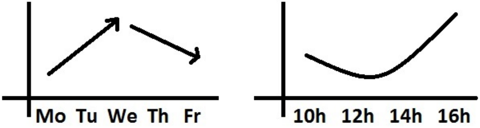
- BAYER AG
- 201404-201409
- 128 trading days
  9:00 - 17:00
- cumulated 1-min volume



Intra-day trading volume

Jain and Joh (1988)

- day of the week and the hour effect the trading volume



Darrat et al. (2003) and Spierdijk et al. (2003)

- lagged values of volatility and trading volume simultaneously

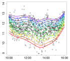
Bialkowski et al. (2008) and Brownlees et al. (2011)

- dynamic volume approach for VWAP

VWAP

Intra-day trading volume

Figure: Expectiles for $\tau \in \{0.01, 0.02, 0.05, 0.1, 0.2, 0.5, , \ldots\}$



Intra-day trading volume

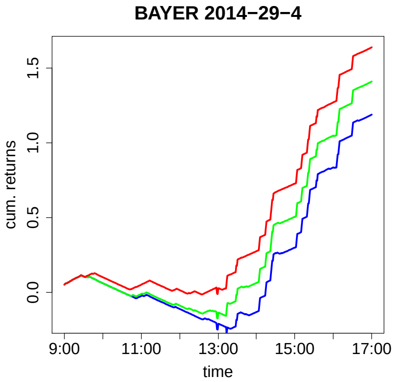
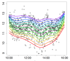


Figure: Cumulative VWAP-returns (trans. costs 0.2%) with weights based on τ-expectiles of volume, $\tau = 0.1, 0.5, 0.9$.



Hurricane predictions

NATIONAL GEOGRAPHIC Daily News

Home Animals Ancient Energy Environment Travel/Cultures Space/Tech Water Weird News Photos News Video News Blogs

Will U.S. Hurricane Forecasting Models Catch Up to Europe's?

A year after Hurricane Sandy, Europe's forecasting technology is still tops.

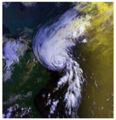
Forbes

INVESTING 4/03/2012 @ 3:50PM | 3,127 views

Damaging Hurricanes Could Impact Energy, Insurance Industries In 2012

Comment Now

by Alan Lammey, Joe Bastardi, Joe D'Aleo, Michael Barak



This image shows Hurricane Bob approaching New England on August 19 at 1226 UTC. This image was produced from data from NOAA-10, provided by NOAA. (Photo credit: Wikipedia)

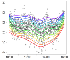
The 2012 Atlantic Hurricane Season is likely to see an overall decrease in tropical activity as compared to 2011, but with the focus of tropical development closer to the United States.

When evaluating the hurricane season as a whole, it is important to consider more data than just the number of storms that are named. Evaluating in multiple ways. The best way to determine total seasonal activity is by collectively measuring the intensity and duration of named tropical cyclones (both tropical storms and hurricanes), also called the ACE index (Accumulated Cyclone Energy). While the National Hurricane

Bloomberg

By Noah Buhayar - Nov 11, 2013

Typhoon Worse for Philippines Economy Than Sandy for U.S.



Trend analysis of tropical storms

- strength of wind
- different trend
- periodicity
  due to sun cycles

▸ Movie of curves

▸ Sun-Hurricane connection

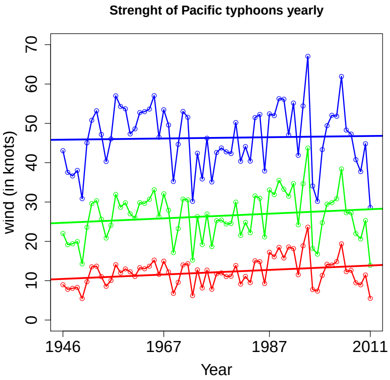
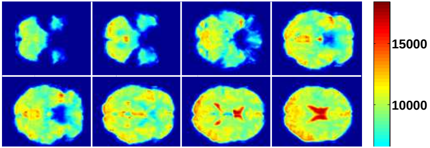


Figure: Annual expectiles for $\tau = 0.25, 0.5, 0.75$ and trend

Joint work with: P.Kokoszka & Q.Xiong (Colorado State University)

fMRI Tail Reaction vs. risk perception

4-point "curves" for each area (256 questions for 19 individuals)



Joint work with: P.Majer (HU)

Temperature data

- Daily average temperature
- Model residuals
  ▸ Model
- Application: Pricing weather derivatives

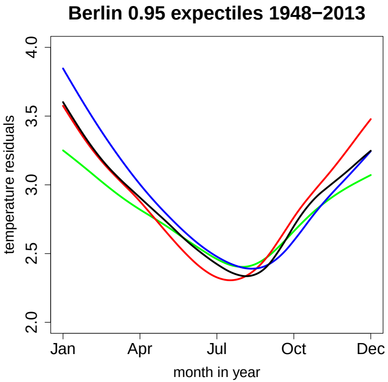
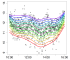
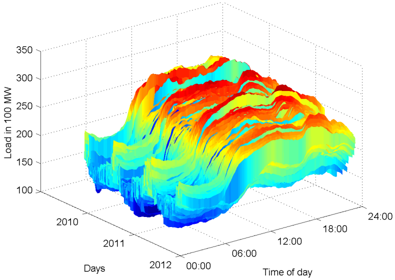


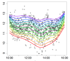
Figure: 0.95-expectile of Berlin temperature residuals in 1948-1969 1970-1991 1992-2013 1948-2013



Dynamic demand models

- **Electricity demand**
  - Quarter-hourly
  - Jan.2010 - Dec.2012
  - Amprion company in west of Germany
- **Water demand**
- **Gas demand**



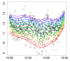


Challenges

- Tail Event Curves (TEC)
- Time-varying TECs
- Reduce dimension
- Dynamics and Dependence
- Forecasts for pricing and many other applications

Outline

1. Motivation ✓
2. Quantiles and Expectiles
3. Modeling Time-varying Curves
4. Outlook
5. Empirical Study
6. References



Quantiles and Expectiles

For r.v. Y obtain tail event measure:

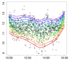
$$q_\tau = \arg\min_\theta \mathsf{E}\left\{\rho_\tau\left(Y - \theta\right)\right\}$$

asymmetric loss function

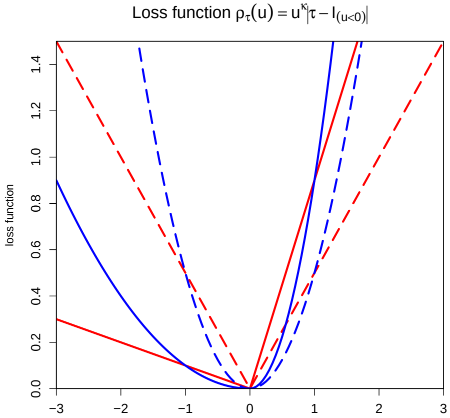
$$\rho_\tau\left(u\right) = \left|u\right|^\alpha \left|\tau - \mathbf{I}_{\{u<0\}}\right|$$

$\alpha = 1$ for quantiles, $\alpha = 2$ for expectiles

Expectile as quantile

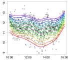


Quantiles and Expectiles



LQRcheck

Figure: Loss function of expectiles and quantiles
for $\tau = 0.5$ (dashed) and $\tau = 0.9$ (solid)



Expectile Curves

Generalized regression τ-expectile

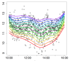
$$e_\tau(x) = \arg\min_\theta \mathsf{E}\left\{\rho_\tau\left(Y - \theta\right) \mid X = x\right\}$$

Expectile $e_\tau(x)$ approximated by B-spline basis Definition of B-splines

Penalized splines:

$$\widehat{\alpha} = \arg\min_\alpha \sum_{i=1}^{n} \rho_\tau \left\{ y_i - \alpha^\top b(x_i) \right\} + \lambda \alpha^\top \Omega \alpha$$

with penalization matrix Ω and shrinkage λ Penalization matrix



Estimation of Expectile Curves

Schnabel and Eilers (2009): iterative LAWS algorithm ▸ LAWS

Schnabel (2011): expectile sheets for joint estimation of curves

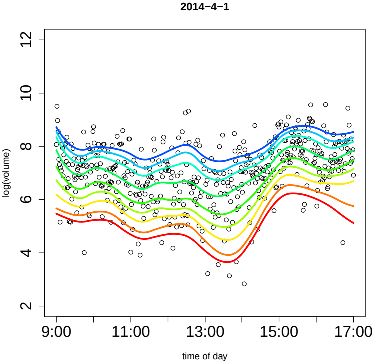


Figure: BAYER trading volume expectiles ($\lambda = 0.5$) on 2014-04-01

Dynamic Tail Event Curves

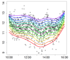
Fix τ :

$$e(t) = \sum_{k=1}^{K} \alpha_k \phi_k(t)$$

with basis $\Phi = (\phi_1, \ldots, \phi_K)^\top$ and $t = 1, \ldots, T$.

Variation in time, $s = 1, \ldots, S$:

$$e_s(t) = \sum_{k=1}^{K} \alpha_{sk} \phi_k(t)$$



Independent curves

Guo et al. (2013)

- $e_s(t)$ **independent** realizations of stationary process
- Karhunen-Loève expansion:

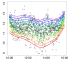
Functional princ. components

$$e_s(t) = \mu(t) + \sum_{k=1}^{K} \alpha_{sk}\phi_k(t) = \mu(t) + \alpha_s^\top \Phi(t)$$

- Penalized splines for approximated mean function and FPC
- Empirical loss function

$$\sum_{s=1}^{S}\sum_{t=1}^{T} \rho_\tau \left\{ Y_{st} - \theta_\mu^\top b(t) - \alpha_s^\top \Theta_\Phi b(t) \right\} + \text{pen.mtx}$$

Details



Temporal (weak) dependent curves

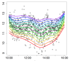
Sequence $\{X_n\}$ is *m*-**dependent** if for any k the σ-algebras $\mathcal{F}_k^- = \sigma\,(\ldots, X_{k-1}, X_k)$ and $\mathcal{F}_{k+m}^+ = \sigma\,(X_{k+m}, X_{k+m+1}, \ldots)$ are independent.

Hörmann and Kokoszka (2010)

- Karhunen-Loève expansion applicable for *m*-dependent
- asymptotic properties of FPC estimates remain the same

- most of time series ARE NOT *m*-dependent
- fail if i.i.d. curves are too noisy
- fail if curves are sufficiently regular but dependency is too strong

How to model stronger dependency?



Cramér-Karhunen-Loève representation

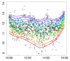
Panaretos and Tavakoli (2013)

- spectral decomposition of stationary functional time series

$$e_s(t) \approx \sum_{j=1}^{J} \exp(\mathrm{i}\omega_j s) \sum_{k=1}^{K} \alpha_{jk}\phi_{jk}(t)$$

$-\pi = \omega_1 < \ldots < \omega_{J+1} = \pi$
$\left\{\phi_{jk}\right\}_{k\geq 1}$ eigenfunctions
$\left\{\alpha_{jk}\right\}_{k\geq 1}$ corresponding coefficients

▸ Theoretical details



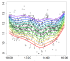
Cramér-Karhunen-Loève representation

Simplify $\phi_{jk}(t) = \phi_k(t)$ for each j

$$
\begin{aligned}
e_s(t) &\approx \sum_{j=1}^{J} \exp(\mathrm{i}\omega_j s) \sum_{k=1}^{K} \alpha_{jk}\phi_k(t) \\
&\approx U(s)^\top A\Phi(t)
\end{aligned}
$$

where

$U(s) = \left(\exp\left(\mathrm{i}\omega_1 s\right), \ldots, \exp\left(\mathrm{i}\omega_J s\right)\right)^\top$
$A_{JxK} = (\alpha_{jk})$ matrix of coefficients
$\Phi(t) = (\phi_1(t), \ldots, \phi_K(t))^\top$



Empirical loss function

Represent $\phi_k(t) = \sum_{l=1}^{L} \beta_{kl} b_l(t)$

$$\Phi(t) = B_{KxL} b(t)$$

Loss function

$$\sum_{s=1}^{S} \sum_{t=1}^{T} \rho_\tau \left\{ Y_{s,t} - U(s)^\top C b(t) \right\} + \lambda \| C_{JxL} \|_{\mathcal{G}_J}$$

where $C_{JxL} = A_{JxK} B_{KxL}$ matrix of coefficients
and $\| C_{JxL} \|_{\mathcal{G}_J}$ Group lasso penalization

$$\| C_{JxL} \|_{\mathcal{G}_J} = \sum_{j=1}^{J} \| c_{\mathcal{G}_j} \|_2 = \sum_{j=1}^{J} \sqrt{\sum_{l=1}^{L} c_{jl}^2}$$

Empirical loss function

$$\underset{C}{\operatorname{argmin}} \underbrace{\sum_{s=1}^{S}\sum_{t=1}^{T}\rho_\tau\left\{Y_{s,t}-U(s)^\top Cb(t)\right\}}_{\mathrm{l(C)}}+\lambda\sum_{j=1}^{J}\|c_{\mathcal{G}_j}\|_2$$

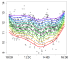
$l(C)$ continuously differentiable

K-K-T conditions for $\widehat{C}$ to be a solution:

$$\nabla l(\widehat{C})_{\mathcal{G}_j}+\lambda\frac{C_{\mathcal{G}_j}}{\|C_{\mathcal{G}_j}\|_2} = 0 \quad \text{if } C_{\mathcal{G}_j}\neq 0$$

$$\|\nabla l(\widehat{C})_{\mathcal{G}_j}\|_2 \leq \lambda \quad \text{if } C_{\mathcal{G}_j}=0$$

If $\tau=0.5$, closed form solution available



Block-Coordinate Gradient Descent Algorithm (BCGD)

Tseng & Yun (2009)

Solve nonconvex nonsmooth optimization problem

$$\min_{x} f(x) + \lambda P(x)$$

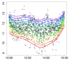
where $\lambda > 0$

$P : \mathbb{R}^n \to (-\infty, \infty]$ block-separable convex function

f smooth on an open subset containing $\mathrm{dom}\, P$

- combination of quadratic approximation and coordinate descent algorithm
- global convergence

▸ Theoretical details



BCGD Algorithm

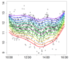
Stepwise and blockwise minimize

$$S_\lambda(\widehat{C}^{(t)}+d) = l(\widehat{C}^{(t)})+d^\top \nabla l(\widehat{C}^{(t)})+\frac{1}{2}d^\top H^{(t)} d+\lambda \sum_{j=1}^{J} \| C_{\mathcal{G}_j}^{(t)}+d\|_2$$

Details

Notation: $\min_{d_{\mathcal{G}_j}} S_\lambda(\widehat{C}^{(t)} + d)$

minimization, where $d = (d_{\mathcal{G}_1}, \ldots, d_{\mathcal{G}_J})$ with $d_{\mathcal{G}_k} = 0$ for $k \neq j$



BCGD Algorithm

repeat

  t=t+1

  for $j = 1$ to J **do**

    if $\|\nabla l(\widehat{C}^{(t)})_{\mathcal{G}_j} - h^{(t)}_{\mathcal{G}_j}\widehat{C}^{(t)}_{\mathcal{G}_j}\|_2 \leq \lambda$ **then**

      $d^{(t)}_{\mathcal{G}_j} = -\widehat{C}^{(t)}_{\mathcal{G}_j}$

    else

      $d^{(t)}_{\mathcal{G}_j} = \min_{d_{\mathcal{G}_j}} S_\lambda(\widehat{C}^{(t)} + d)$

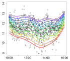
    end if

  end for

until convergence criterion met

update $\widehat{C}^{(t+1)} - \widehat{C}^{(t)} - \alpha^{(t)} d^{(t)} = 0$

Details



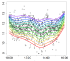
DYTEC Algorithm

Start with initial weights $w_{s,t}$ (obtained separately for each $s = 1, \ldots, S$) and iterate between following steps:

- compute $\widehat{C}$ using BCGD algorithm
- update weights

$$w_{s,t} = \begin{cases} \tau & \text{if } Y_{s,t} > U(s)^\top \widehat{C} b(t), \\ 1 - \tau & \text{otherwise.} \end{cases}$$

- stop if there is no change in weights $w_{s,t}$.



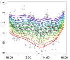
Dynamic functional factor model

- generalization (capture nonstationarity)
- extend with the idea of two spaces of basis function

Model

$$e_s(t) = \sum_{k=1}^{K} Z_{sk} m_k(t) = Z_s^\top m(t)$$

with time-varying factor loadings Z_k
and functional factors $m_k(t)$



Dynamic functional factor model

Space basis:

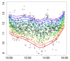
$$m_k(t) = \sum_{l=1}^{L} \beta_{kl} b_l(t)$$

$$m(t) = B_{KxL} b(t)$$

Time basis:

$$Z_{sk} = \sum_{j=1}^{J} \alpha_{kj} u_j(s)$$

$$Z_s = A_{KxJ} \cdot U(s)$$



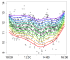
Dynamic functional factor model

$$e_s(t) = Z_s^\top m(t) = U(s)^\top C b(t)$$

with $C_{JxL} = A_{JxK}^\top B_{KxL}$ matrix of coefficients,
space basis vector $b(t) = \{b_1(t), \ldots, b_L(t)\}^\top$
and time basis vector $U(s) = \{u_1(s), \ldots, u_J(s)\}^\top$

Same loss function:

$$\sum_{s=1}^{S} \sum_{t=1}^{T} \rho_\tau \left\{ Y_{s,t} - U(s)^\top C b(t) \right\} + \lambda \| C_{JxL} \|_{\mathcal{G}_J}$$

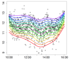


Time basis

- capture periodic variation
- capture trend

- Proposal by Song et al. (2013):
  - Legendre polynomial basis
  - Fourier series

See Appendix

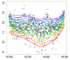


Space basis

- capture daily patterns
- capture specific structure

- Proposal by Song et al (2013):
  - Data driven
  - Based on combination of smoothing techniques and FPCA
- B-splines

Definition of B-splines



Empirical Study

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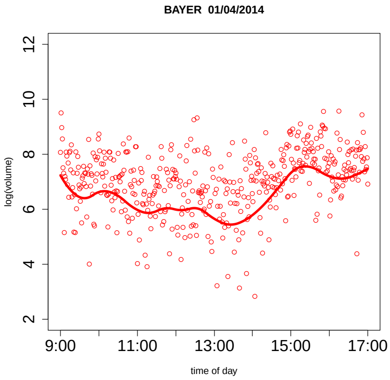
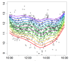
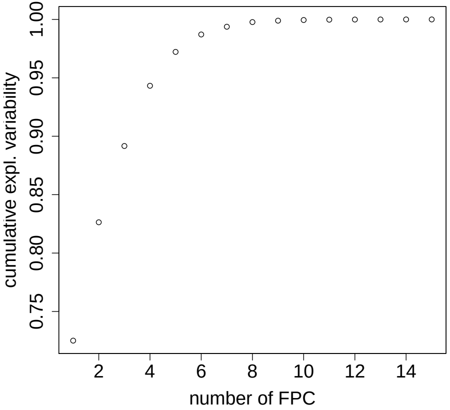


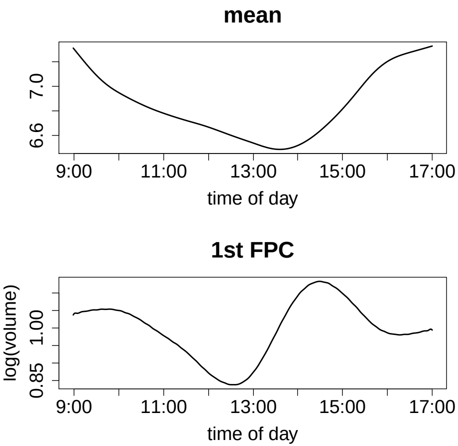


Trading volume - FPCA

Use 4 FPCs to explain 90% of variation



Trading volume - FPCA



Periodicity of 1st PC score

Fisher's G-test: p-value=0.0002

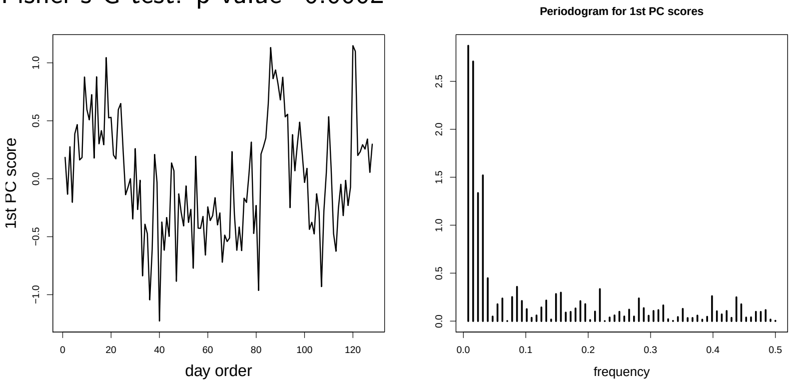


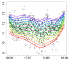
Figure: Scores of 1FPC and periodogram

Outlook

- Dynamic model for tail event curves
- Considering pattern and dependency
- Provide forecasts

TBD:

- Algorithm - Code in R
- Simulation and Empirical studies



DYTEC - DYnamic Tail Event Curves and its applications

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http://case.hu-berlin.de

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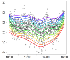

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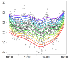

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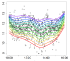

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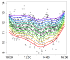

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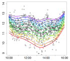

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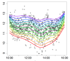

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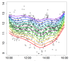


VWAP trading strategy

- Buying/Selling fixed amount of shares at average price $p_d(t)$ that tracks the VWAP-benchmark of that day

$$VWAP_d^\tau = \frac{\sum_{t=1}^{T} v_{d-1}^\tau(t) \cdot p_{d-1}(t)}{\sum_{t=1}^{T} v_{d-1}^\tau(t)}$$

where $v_d^\tau(t)$ is τ-expectile of 1-min cumulated volume

and $p_d(t)$ price at time t on day d

- 50 % of trades are VWAP orders
- Implementation requires **model for intraday evolution of volume**

Back to Motivation - VWAP

Expectiles Curves for Typhoons

Figure: Every year with expectile curves for $\tau = 0.1, 0.5$ and 0.9.

Back to Motivation

Expectiles Curves for Hurricanes

Figure: Every year with expectile curves for $\tau = 0.1, 0.5$ and 0.9.

Back to Motivation

Number of Hurricanes

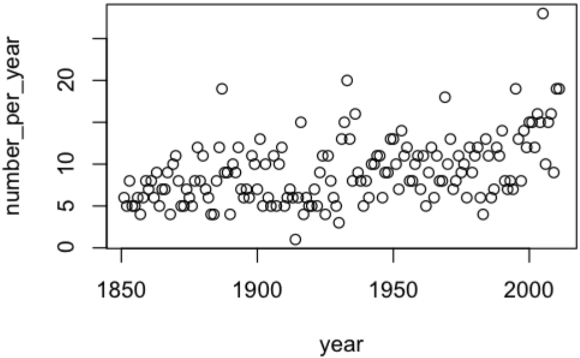


Figure: Yearly number of hurricanes

Back to Motivation

Test of trend

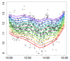
$$e_s^{\tau}(t) = \alpha_{\tau}(t) + s\beta_{\tau}(t) + \varepsilon_{\tau}(t)$$

Test for different τ

$$H_0 : \beta(t) = 0$$

Result: do not reject for all τ
(p-values in range 0.263-0.279)

Back to Motivation



Typhoon's curves differences

$e_n^\tau(t)$—τ-expectile curve for year n

$$\hat{\mu}_k^\tau(t) = \frac{1}{k}\sum_{n=1}^{k} e_n^\tau(t)$$

$$\tilde{\mu}_k^\tau(t) = \frac{1}{N-k}\sum_{n=k+1}^{N} e_n^\tau(t)$$

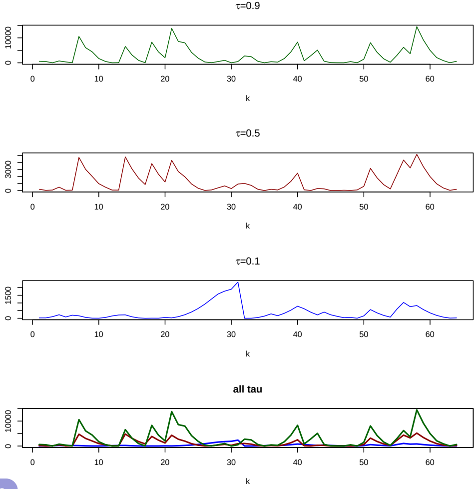
Normalized differences:

$$P_{\tau,k}(t) = \frac{k(N-k)}{N}\left\{\hat{\mu}_k^\tau(t) - \tilde{\mu}_k^\tau(t)\right\}$$

$$P_{\tau,k} = \int_0^T P_{\tau,k}^2(t)dt$$

Back to Motivation

Typhoon's curves differences



Back to Motivation

Sun-Typhoons connection

Elsner,Jagger (2008)

- United States and caribbean tropical cyclone activity related to the colar cycle

Hodges, Jagger, Elsner (2014)

- The sun-hurricane connection: Diagnosing the solar impacts on hurricane frequency over the North Atlantic basin using a spaceâtime model

Guy Carpenter& Company (Sep. 2014)

- The Hurricane Seasons that Changed the Insurance Industry

▸ Back to Motivation

Model of Temperature data

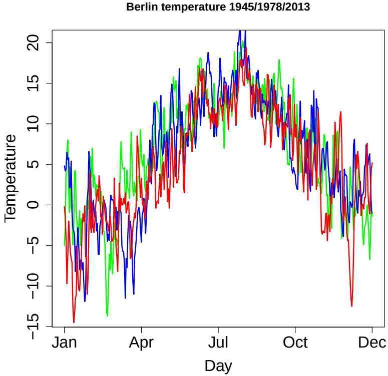
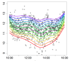


Figure: Daily average temperature in Berlin in 1948, 1980, 2013

Back to Motivation - Temperature Data



Model of Temperature data

For days $t = 1, \ldots, 24090$ (i.e. 66 years)

$$X_t = T_t - \Lambda_t$$

$$\Lambda_t = a + bt + \sum_{m=1}^{2} \left\{ c_m \cos\left(\frac{m \cdot \pi \cdot t}{365}\right) + d_m \sin\left(\frac{m \cdot \pi \cdot t}{365}\right) \right\}$$

$$X_t = \sum_{l=1}^{L} \beta_l X_{t-l} + \varepsilon_t$$

Back to Motivation - Temperature Data

Model of Temperature data

$$\hat{a} = 5.617 \qquad \hat{\beta}_1 = 0.786$$

$$\hat{b} = 2.3 * 10^{-5} \qquad \hat{\beta}_2 = -0.078$$

$$\hat{c}_1 = -4.15 * 10^{-2} \qquad \hat{\beta}_3 = 0.024$$

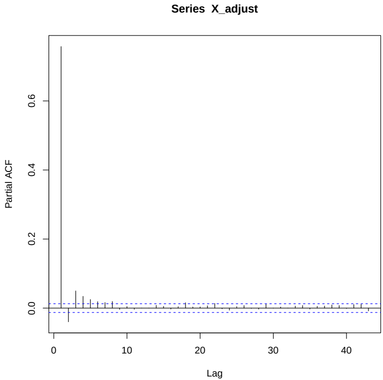
$$\hat{c}_2 = -7.14 * 10^{-2} \qquad \hat{\beta}_4 = 0.015$$

$$\hat{d}_1 = -7.932 \qquad \hat{\beta}_5 = 0.011$$

$$\hat{d}_2 = -3.109 \qquad \hat{\beta}_6 = 0.007$$

$$\hat{\beta}_7 = 0.001$$

$$\hat{\beta}_8 = 0.019$$



Back to Motivation - Temperature Data

Relating Expectiles and Expected Shortfall

Newey and Powell (1987):

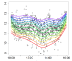
$$e_\tau = \arg\min_e \mathsf{E}\left\{\left|\tau - \mathbf{I}_{\{Y<e\}}\right|(Y-e)^2\right\}$$

$$\frac{1-2\tau}{\tau}\mathsf{E}\left\{(Y-e_\tau)\,\mathbf{I}_{\{Y<e_\tau\}}\right\} = e_\tau - \mathsf{E}(Y)$$

Taylor (2008):

$$\mathsf{E}\left(Y|Y<e_\tau\right) = e_\tau + \frac{\tau\left\{e_\tau - \mathsf{E}(Y)\right\}}{(1-2\tau)F(e_\tau)}$$

Back



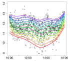
Expectile as quantile

$e_\tau(Y)$ is the τ-quantile of the cdf T, where

$$T(y) = \frac{G(y) - xF(y)}{2\{G(y) - yF(y)\} + \{y - \mu_Y\}}, \tag{1}$$

$$G(y) = \int_{-\infty}^{y} u \, dF(u) \tag{2}$$

Back Quantiles and Expectiles



B-splines

Knot vector $t = (t_1, \dots, t_M)$ as nondecreasing sequence in $[0, 1]$

Control points $P_0, \dots, P_N$

Define i-th B-spline basis function $N_{i,j}$ of order j as

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i < t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,j}(t) = \frac{t - t_i}{t_{i+j} - t_i} N_{i,j-1}(t) + \frac{t_{i+j+1} - t}{t_{i+j+1} - t_{i+1}} N_{i+1,j-1}(t)$$

$j = 1, \dots, N - M - 1$

‣ Back to Estimation of Expectile Curves

‣ Back to Space Basis

Quantile Curves Penalty

$$\widehat{\alpha} = \arg\min_{\alpha} \sum_{i=1}^{n} \rho_\tau \left\{ y_i - \alpha^\top b(x_i) \right\} + \lambda \alpha^\top \Omega \alpha$$

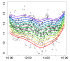
where $b(x) = (b_1, \ldots, b_K)^\top$ is vector of B-spline basis functions

Denote $\tilde{b}(x) = (\tilde{b}_1(x), \ldots, \tilde{b}_K(x))^\top$ the vector of second derivatives of basis functions

and set

$$\Omega = \int \tilde{b}(x) \tilde{b}(x)^\top \, dx$$

▸ Back to Estimation of Expectile Curves



LAWS estimation

Schnabel and Eilers (2009):

$$\min \sum_{i=1}^{n} w_i(\tau)(y_i - \mu_i)^2$$

where

$$w_i(\tau) = \begin{cases} \tau & \text{if } y_i > \mu_i \\ 1 - \tau & \text{if } y_i \leq \mu_i, \end{cases}$$

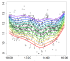
μ_i expected value according to some model.

Iterations:

- fixed weights, closed form solution of weighted regression
- recalculate weights

until convergence criterion met.

Back to Expectiles Curves



LAWS estimation

Example:

Classical linear regression model

$$Y = X\beta + \varepsilon$$

where $\mathsf{E}(\varepsilon \,|X) = 0$ and $\mu = \mathsf{E}(Y\,|X) = X\beta$.

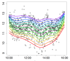
$$\widehat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} w_i (y_i - \mu_i)^2$$

Then:

$$\widehat{\beta} = (X^\top W X)^{-1} X W Y$$

with W diagonal matrix of fixed weights w_i.

Back to Expectile Curves



Functional principal components

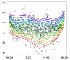
$X(t)$ stochastic process on compound interval T
with mean function $\mu(t) = \mathrm{E}\{X(t)\}$
and covariance function $k(s,t) = \mathrm{cov}(X(s), X(t))$

There exist orthogonal sequence of eigenfunctions ϕ_j and eigenvalues λ_i such that $k(s,t) = \sum_{j=1}^{\infty} \lambda_j \phi_j(s)\phi_j(t)$
We can rewrite process as

$$X(t) = \mu(t) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\kappa_j\phi_j(t)$$

where $\kappa_j = \frac{1}{\sqrt{\lambda_j}} \int X(t)\phi_j(s)ds$, $\mathrm{E}(\kappa_j) = 0$ and $\mathrm{E}(\kappa_j\kappa_K) = \delta_{jk}$.

Back to Modelling of Time-varying Curves



Guo(2013) - Empirical loss function

$$S^* = S + M_\mu + M_\Phi$$

where

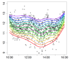
$$S = \sum_{d=1}^{D} \sum_{t=1}^{T} \rho_\tau \left\{ Y_{dt} - b(t)^\top \theta_\mu - b(t)^\top \Theta_\Phi \alpha_d \right\}$$

$$M_\mu = \theta_\mu^\top \int \tilde{b}(x)\tilde{b}(x)^\top \, dx \theta_\mu = \theta_\mu^\top \Omega \theta_\mu$$

$$M_\Phi = \sum_{k=1}^{K} \theta_{\phi,k} \int \tilde{b}(x)\tilde{b}(x)^\top \, dx \, \theta_{\phi,k}$$

and $\tilde{b}(x)$ vector of second derivatives

▸ Back to Guo(2013) - Estimation of Quantile Curves



Cramér-Karhunen-Loève representation

Conditions (Panaretos and Tavakoli (2013))

X_t second order stationary time series in $L^2([0,1],\mathbb{R})$
with zero mean, $\mathrm{E}\,\|X_0\|_2^2 < \infty$ and autocovariance kernel at lag t :

$$r_t(u,v) = \mathrm{E}\,\{X_t(u)X_0(v)\}$$

$u, v \in [0,1],\ t \in \mathbb{Z},$ inducing operator :

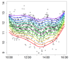
$$\mathcal{R}_t : L^2([0,1],\mathbb{R}) \to L^2([0,1],\mathbb{R})$$

Assume:

i) $\sum_{t\in\mathbb{Z}} \|\mathcal{R}_t\|_1 < \infty$

ii) $(u,v) \to r_t(u,v)$ continuous $t \in \mathbb{Z}$, and $\sum_{t\in\mathbb{Z}} \|r_t\|_\infty < \infty$

Back to C-K-L representation



Cramér-Karhunen-Loève representation

Theorem (Panaretos and Tavakoli (2013))

X_t admits representation

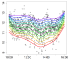
$$X_t = \int_{-\pi}^{\pi} \exp(\mathrm{i}\omega_j t) dZ_\omega \text{ a.s.}$$

where for fixed ω, Z_ω is random element of $\mathrm{L}^2([0,1], \mathbb{C})$
and process $\omega \to Z_\omega$ has orthogonal increments.

Integral can be understood as a Riemann-Stieltjes limit in sense

$$\mathrm{E}\, \|X_t - \sum_{j=1}^{J} \exp(\mathrm{i}\omega_j t)(Z_{\omega_{j+1}} - Z_{\omega_j})\|_2^2 \to 0 \text{ as } J \to 0$$

▸ Back to C-K-L representation



Cramér-Karhunen-Loève representation

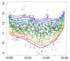
Remark (Panaretos and Tavakoli (2013))

With spectral density operator $\mathcal{F}_\omega = \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \exp(-\mathrm{i}\omega t) \mathcal{R}_t$ having eigenfunctions $\{\phi_n^\omega\}_{t \geq 1}$

C-K-L representation can be interpreted as

$$X_t = \int_{-\pi}^{\pi} \exp(\mathrm{i}) \sum_{n=1}^{\infty} \langle \phi_n^\omega, dZ_\omega \rangle \, \phi_n^\omega$$

Back to C-K-L representation



Block-Coordinate Gradient Descent Algorithm

Tseng & Yun (2009)

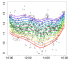
$$\min_x f(x) + \lambda P(x)$$

Solve

$$\min_d \left\{ d^\top \nabla f(x) + \frac{1}{2} d^\top H d + \lambda P(x+d) \right\}$$

- P is block-separable then H is block-diagonal
- solve subproblems
  (for every j take $d = (d_{\mathcal{G}_1}, \ldots, d_{\mathcal{G}_J})$ with $d_{\mathcal{G}_k} = 0$ for $k \neq j$)

Back to BCGD algorithm



Block-Coordinate Gradient Descent Algorithm

$$d_{\mathcal{G}_j}^{(t)} = -\frac{1}{h_{\mathcal{G}_j}^{(t)}} \left(\nabla l(\widehat{C}^{(t)})_{\mathcal{G}_j} - \lambda \frac{\nabla l(\widehat{C}^{(t)})_{\mathcal{G}_j} - h_{\mathcal{G}_j}^{(t)} \widehat{C}_{\mathcal{G}_j}^{(t)}}{\|\nabla l(\widehat{C}^{(t)})_{\mathcal{G}_j} - h_{\mathcal{G}_j}^{(t)} \widehat{C}_{\mathcal{G}_j}^{(t)}\|_2} \right)$$

$H^{(t)}$ has submatrices $H_{\mathcal{G}_j}^{(t)} = h_{\mathcal{G}_j}^{(t)} I_{\mathcal{G}_j}$ for scalars $h_{\mathcal{G}_j}^{(t)}$

$\alpha^{(t)}$ set by Amijo rule (See Details: Tseng & Yun (2009))

Back to BCGD algorithm

Song et. al.(2013) - Time basis

Orthogonal Legendre polynomial basis
to capture the global trend in time
$u_1(d) = 1/C_1,\ u_2(d) = d/C_2,\ u_3(d) = (3t^2 - 1)/C_3, \ldots$
with generic constant C_i such that $\sum_{d=1}^{D} \mathrm{u}_i(d)/C_i^2 = 1$

Fourier series
to capture periodic variations
$u_4 = \sin(2\pi d/p)/C_4,\ u_5 = \cos(2\pi d/p)/C_5,$
$u_6 = \sin(2\pi d/(p/2))/C_6, \ldots$
with given period p

Back to Song(2013) -Time Basis