Conditional Systemic Risk with Penalized Copula

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Contagion and systemic risk measures

- Connectedness measures from volatility, Diebold and Yilmaz (2014, JoE).
- Credit risk
  - Factor/Copula models, Cherubini and Mulinacci (2015).
  - Econometric models, Lucas et al. (2014, JBES).
- Expected shortfall, Acharya et al. (2010) and Brownlees and Engle (2012).

Conditional quantile-based measures

- CoVaR and ΔCoVaR, Adrian and Brunnermeier (2011).
- Properties of CoVaR, Mainik and Schaanning (2014, SRM).
- Large "p" and linear quantiles, Hautsch et al. (2014, RoF).
- Large "p" and non-linear quantiles, Härdle et al. (2015).
- CAViaR, Engle and Manganelli (2004, JBES).
- VAR for VaR, White et al. (2015, JoE) .
- ...

Contribution

- Consistent framework to measure contagion/systemic risk.
  - No structural assumptions on conditional quantile!
  - Bivariate relations, sub-portfolios, systemic analysis.
  - Intuitive properties and simple interpretation.
- Sparse Hierarchical Archimedean Copula (HAC).
  - Few parameters.
  - Flexible dependence in tail area.

Outline

1. Motivation ✓
2. Contagion and Systemic Risk
3. Penalized Hierarchical Archimedean Copula
4. Simulation
5. Application
6. Summary

Conditional quantile

- Two rv X_k and X_ℓ with joint cdf $F(x_k, x_\ell)$ and conditional cdf

$$F_{X_k|X_\ell = x_\ell}(x_k) = \mathrm{P}(X_k \leq x_k | X_\ell = x_\ell).$$

- Conditional quantile, $\alpha \in (0, 1)$,

$$Q_{X_k|X_\ell = x_\ell}(\alpha) = F^{-1}_{X_k|X_\ell = x_\ell}(\alpha).$$

- Unconditional margins
  - $u_j = F_j(x_j)$ and $Q_j(\alpha) = F_j^{-1}(\alpha)$,
  - $U_j = F_j(X_j)$ and $U_j \sim \mathrm{U}(0, 1)$, $j = k, \ell$.

Conditional quantile and copula

- Thanks to Sklar (1959) $F(x_k, x_\ell) = C\{F_k(x_k), F_\ell(x_\ell)\}$.
- Conditional copula

$$C_{U_k|U_{\not{k}}=u_{\not{k}}}(u_k) = \mathrm{P}(U_k \leq u_k | U_\ell = u_\ell).$$

- C-quantiles, c.f. Bouyé and Salmon (2009, EJoF), $\alpha \in (0,1)$,

$$Q_{X_k|X_{\not{k}}=x_{\not{k}}}(\alpha) = Q_k\{C^{-1}_{U_k|U_{\not{k}}=u_{\not{k}}}(\alpha)\} = Q_{X_k|U_{\not{k}}=u_{\not{k}}}(\alpha).$$

Conditional quantile does not depend on the law of X_ℓ.

Partial effects

- With density $f_j(x_j) = F_j'(x_j)$ and quantile density $q_j(\alpha) = Q_j'(\alpha)$, $j = k, \ell$, see Parzen (1979, JASA),

$$\frac{\partial}{\partial x_\ell} Q_{X_k|X_{\not{k}}=x_{\not{k}}}(\alpha) = \frac{q_k\{C^{-1}_{U_k|U_{\not{k}}=u_{\not{k}}}(\alpha)\}}{q_\ell(u_\ell)} \frac{\partial}{\partial u_\ell} C^{-1}_{U_k|U_{\not{k}}=u_{\not{k}}}(\alpha).$$

Partial derivative depends on law of X_ℓ as

$$q_\ell(\alpha) = \frac{1}{f_\ell\{Q_\ell(\alpha)\}}.$$

Contagion

- Contagion to k from ℓ as normalized partial effect

$$\mathcal{S}_{k\ell}^{u_{\not k}} \stackrel{\text{def}}{=} \frac{Q_\ell(u_\ell) q_k\{C^{-1}_{U_k|U_{\not k}=u_{\not k}}(\alpha)\}}{q_\ell(u_\ell) Q_k\{C^{-1}_{U_k|U_{\not k}=u_{\not k}}(\alpha)\}} \frac{\partial}{\partial u_\ell} C^{-1}_{U_k|U_{\not k}=u_{\not k}}(\alpha).$$

- At level $\alpha \in (0,1)$, $\mathcal{S}_{k\ell}^{\alpha} = \mathcal{S}_{k\ell}^{u_{\not k}}\Big|_{u_{\not k}=\alpha}$.
- Import interpretation of elasticities from economics, see Sydsæter and Hammond (1995).

Contagion

- Contagion to k from ℓ as normalized partial effect

$$\mathcal{S}_{k\ell}^{u_{\not k}} \stackrel{\text{def}}{=} \frac{x_\ell}{Q_{X_k|X_{\not k}=x_{\not k}}(\alpha)} \frac{\partial}{\partial x_\ell} Q_{X_k|X_{\not k}=x_{\not k}}(\alpha).$$

- At level $\alpha \in (0,1)$, $\mathcal{S}_{k\ell}^{\alpha} = \mathcal{S}_{k\ell}^{u_{\not k}}\Big|_{u_{\not k}=\alpha}$.
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Interpretation

- If $|\mathcal{S}_{k\ell}^{\alpha}| \approx \infty$, $Q_{X_k|U_{\not{k}}=\alpha}(\alpha)$ is sensitive wrt to changes in x_ℓ.
- If $|\mathcal{S}_{k\ell}^{\alpha}| \approx 1$, $Q_{X_k|U_{\not{k}}=\alpha}(\alpha)$ behaves proportional . . . .
- If $|\mathcal{S}_{k\ell}^{\alpha}| \approx 0$, $Q_{X_k|U_{\not{k}}=\alpha}(\alpha)$ is robust . . . .

Asymmetric matrix $\{\mathcal{S}_{k\ell}^{\alpha}\}_{k,\ell=1}^{d}$. If $\mathcal{S}_{k\ell}^{\alpha}$ and $\mathcal{S}_{\ell k}^{\alpha}$ . . .

- . . . have a positive sign, risks are substitutes.
- . . . have a negative sign, risks are complements.
- . . . have a different sign, no statement can be made.

Studying tail areas

- Conditional tail independence, c.f. Bernard and Czado (2015, JMVA)
  - X_k and X_ℓ are called conditionally independent in the right tail if $\lim_{x_\ell \to \infty} Q_{X_k | X_\ell = x_\ell}(\alpha) = g(\alpha)$, $\alpha \in (0,1)$, with $g(\cdot)$ independent of x_ℓ.
- Tail-monotonicity, c.f. Parzen (1979, JASA) Definition
  - If $f(x)$ is tail-monotone density, then $q(u) \sim (1-u)^{-\gamma}$ as $u \to 1$, with tail exponent $\gamma > 0$.

Proposition

Let X_k and X_ℓ have tail-monotone densities $f_k(x_k)$ and $f_\ell(x_\ell)$ with tail exponents γ_k and γ_ℓ.

(a) *If X_k and X_ℓ are conditionally positive dependent, with $\gamma_k \geq 1$ and $\gamma_\ell > 1$, then $\mathcal{S}_{k\ell}^{u_\ell} \to \frac{\gamma_k - 1}{\gamma_\ell - 1}$ as $u_\ell \to 1$.*

(b) *If X_k and X_ℓ are conditionally positive dependent, with $\gamma_k > 1$ and $\gamma_\ell = 1$, then $\mathcal{S}_{k\ell}^{u_\ell} \to \infty$ as $u_\ell \to 1$.*

(c) *If X_k and X_ℓ are conditionally independent, with $\gamma_k \geq 1$ and $\gamma_\ell \geq 1$, then $\mathcal{S}_{k\ell}^{u_\ell} \to 0$ as $u_\ell \to 1$.*

Heterogenous margins

Example

- Assume $X_k \sim \mathrm{N}(0,3)$ and $X_\ell \sim t_3$, so that $|Q_k(u)| < |Q_\ell(u)|$ for small u
-

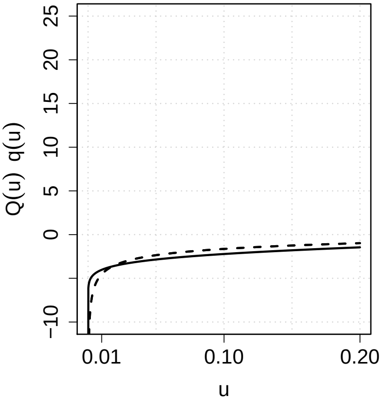

Figure 1: Quantile functions $Q_k(u)$ (solid $\mathrm{N}(0,3)$) and $Q_\ell(u)$ (dashed t_3).

Heterogenous margins

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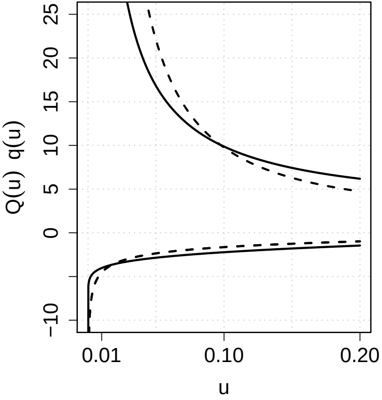


Figure 2: Quantile and quantile density functions $Q_k(u)$, $q_k(u)$ (solid $\mathrm{N}(0,3)$) and $Q_\ell(u)$, $q_\ell(u)$ (dashed t_3).

Heterogenous margins

Example

- Assume $X_k \sim \mathrm{N}(0,3)$ and $X_\ell \sim t_3$, so that $|Q_k(u)| < |Q_\ell(u)|$ and $q_k(u) < q_\ell(u)$ for small u.
- Let $\{F_k(X_k), F_\ell(X_\ell)\}^\top \sim C(u_k, u_\ell; \theta)$, where $C(u_k, u_\ell; \theta)$ refers to the Clayton copula, $\theta = 2$.

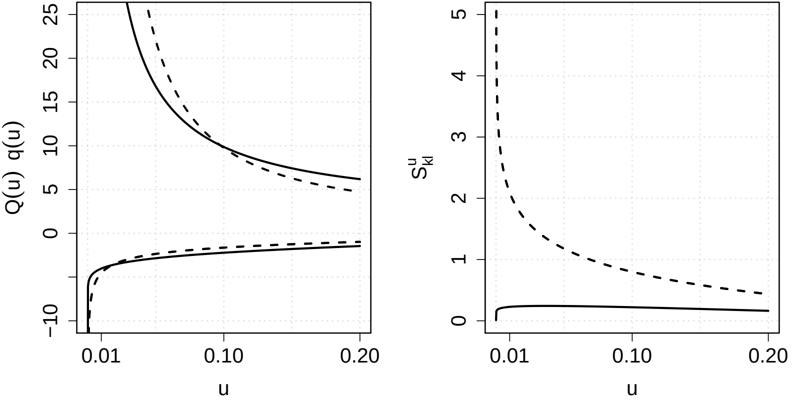


Figure 3: Quantile and quantile density functions $Q_k(u)$, $q_k(u)$ (solid N(0, 3)), $Q_\ell(u)$, $q_\ell(u)$ (dashed t_3) and contagion measures $\mathcal{S}^u_{k\ell}$ (solid) and $\mathcal{S}^u_{\ell k}$ (dashed).

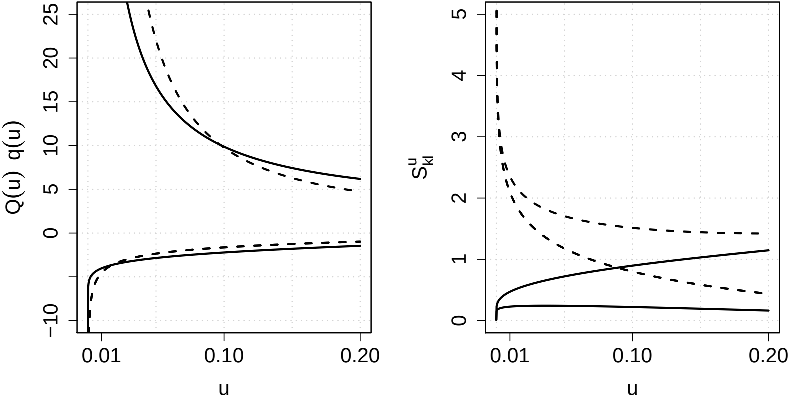


Figure 4: Quantile and quantile density functions $Q_k(u)$, $q_k(u)$ (solid N(0, 3)), $Q_\ell(u)$, $q_\ell(u)$ (dashed t_3) and contagion measures $\mathcal{S}^u_{k\ell}$ (solid) and $\mathcal{S}^u_{\ell k}$ (dashed). ▸ c-quantile

Interpretation

If financial markets k and ℓ with risk factors X_k and X_ℓ are under distress,

- low-risk market is unaffected by increased distress in high-risk market.
- changes in low-risk market imply significant changes in high-risk market, which amplifies a crisis.

Moving to higher dimensions

- Given $X_1, \dots, X_d$ the conditional quantile of X_k

$$Q_{X_k | X_{\not k} = x_{\not k}}(\alpha) = F^{-1}_{X_k | X_{\not k} = x_{\not k}}(\alpha) \quad \text{with} \quad \alpha \in (0,1),$$

  where $\{X_{\not k} = x_{\not k}\}$ refers to event $\{X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_{k+1} = x_{k+1}, \dots, X_d = x_d\}$.

- For normalization
  - $Q_{\not k}(\alpha) = \{Q_1(\alpha), \dots, Q_{k-1}(\alpha), Q_{k+1}(\alpha), \dots, Q_d(\alpha)\}^\top$
  - Define $\|v\| \stackrel{\text{def}}{=} \sqrt[q]{\sum_{j=1}^q v_j^q}$, where q is # of components of v.

Contagion to sub-portfolio

- Contagion to $\mathcal{K}_\ell = \{1, \ldots, d\} \setminus \ell$ from ℓ measured by

$$\mathcal{S}^{\alpha}_{\mathcal{K}_\ell \leftarrow \ell} \stackrel{\text{def}}{=} \frac{\sum_{k \in \mathcal{K}_\ell} Q_{X_k | U_{\not{k}} = \boldsymbol{\alpha}}(\alpha) \mathcal{S}^{\alpha}_{k\ell}}{\sum_{k \in \mathcal{K}_\ell} Q_{X_k | U_{\not{k}} = \boldsymbol{\alpha}}(\alpha)}.$$

- "Diversification" is taken into account.
- AB (2011) interpretation: Pollution of the financial system by institution ℓ given $X_{\not{k}} = Q_{\not{k}}(\alpha)$.

Contagion from sub-portfolio

- Contagion from $\mathcal{L}_k = \{1, \ldots, d\} \setminus k$ to k measured by

$$\mathcal{S}^{\alpha}_{k \leftarrow \mathcal{L}_k} \stackrel{\text{def}}{=} \frac{1}{\| \mathrm{p}_{\not k} \| \| Q_{\not k}(\alpha) \|_2} \sum_{\ell \in \mathcal{L}_k} \mathcal{S}^{\alpha}_{k\ell},$$

  where $\mathrm{p}_{\not k} = (p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_d)^\top$, $p_\ell = 1$ for $\ell \in \mathcal{L}_k$.
- AB (2011) interpretation: Extent institution X_k is affected in case of systemic events.
- Similar to joint shock in factor models.

Systemic risk

- Aggregated effect of "leave-one-out" portfolios.
- $Q(\alpha) = \{Q_1(\alpha), \ldots, Q_d(\alpha)\}^\top$
- p is d-dimensional vector of 1's, so that $\| \mathrm{p} \| = d^{1/d}$.
- Systemic risk is measured by

$$\mathcal{S}^\alpha \stackrel{\text{def}}{=} \frac{1}{\| \mathrm{p} \| \| Q(\alpha) \|_2} \frac{\sum_{k,\ell=1}^{d} Q_{X_k | U_{\not k} = \boldsymbol{\alpha}}(\alpha) \mathcal{S}^\alpha_{k\ell}}{(d-1) \sum_{k=1}^{d} Q_{X_k | U_{\not k} = \boldsymbol{\alpha}}(\alpha)}.$$

Copula families

- Gaussian copula
  - No tail dependence and correlation matrix.
- t-copula
  - One parameter for all tail areas plus correlation matrix.
- Factor copula, Oh and Patton (2014)
  - Flexible, but no density/conditional quantile.
- Vines, Kurowicka and Joe (2011)
  - Flexible, but need $d(d-1)/2$ parameters.
- HAC, Okhrin et al. (2013, SRM)
  - Modelling bias, but few parameters and "flexible" tail dependence.

Archimedean copula

Definition (Multivariate Archimedean copula)

A d-dimensional Archimedean copula $C : [0,1]^d \to [0,1]$ is defined as

$$C(u_1, \ldots, u_d) = \phi\left\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\right\},$$

where $\phi : [0,\infty) \to [0,1]$ is a completely monotone Archimedean copula generator with $\phi(0) = 1$, $\phi(\infty) = 0$.

Example 1

Family	$\phi(u,\theta)$	Parameter range	Independence
Gumbel	$\exp\left(u^{1/\theta}\right)$	$\theta \in [1,\infty)$	$\theta = 1$
Clayton	$(u+1)^{-1/\theta}$	$\theta \in (0,\infty)$	–

Gumbel, Emil Julius on BBI:

Hierarchical Archimedean copula

Example 2

$$C\left(u_1, u_2, u_3; \theta\right) = \phi_{\theta_{(12)3}}\left[\phi^{-1}_{\theta_{(12)3}} \circ \phi_{\theta_{12}}\left\{\phi^{-1}_{\theta_{12}}\left(u_1\right) + \phi^{-1}_{\theta_{12}}\left(u_2\right)\right\} + \phi^{-1}_{\theta_{(12)3}}\left(u_3\right)\right]$$

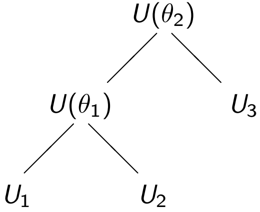


Figure 5: Structure of 3-dim fully nested HAC.

Example 3

$$C(u_1,\ldots,u_4;\theta) = \phi_{(12)(34)}[\phi^{-1}_{(12)(34)} \circ \phi_{12}\left\{\phi^{-1}_{12}(u_1) + \phi^{-1}_{12}(u_2)\right\} + \phi^{-1}_{(12)(34)} \circ \phi_{34}\left\{\phi^{-1}_{34}(u_3) + \phi^{-1}_{34}(u_4)\right\}]$$

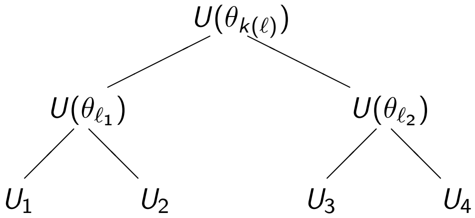
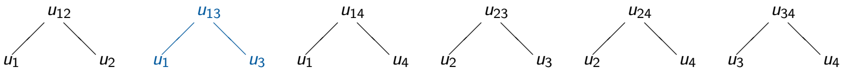
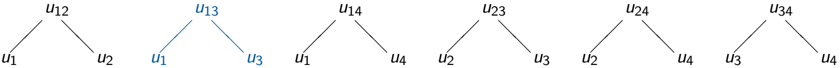


Figure 6: Structure of 4-dim partially nested HAC.

Estimation of HAC

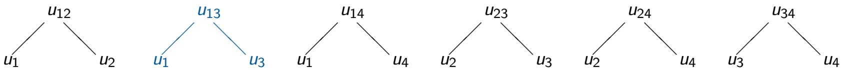


Estimation of HAC



$$\max\{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{34}\} = \hat{\theta}_{13} \quad \Rightarrow$$

Estimation of HAC



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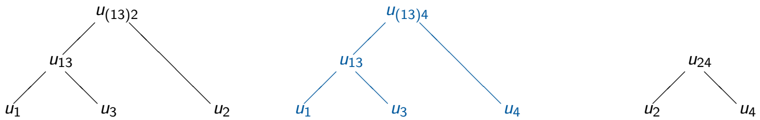

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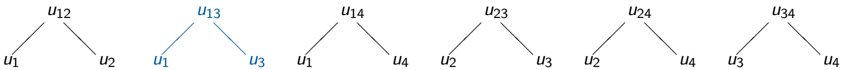

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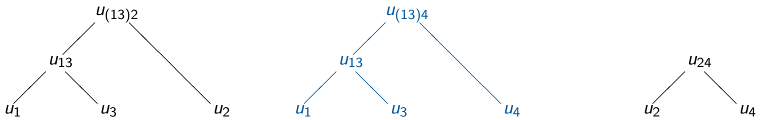
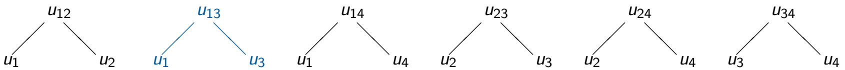


$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4} \quad \Rightarrow$$

Estimation of HAC



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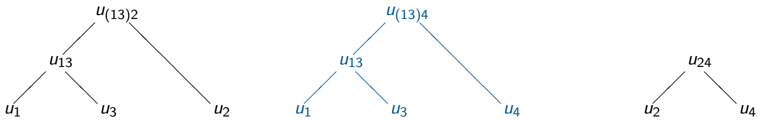

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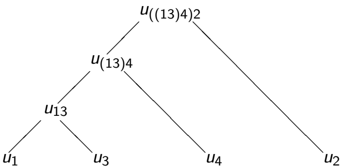

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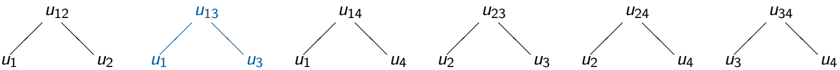

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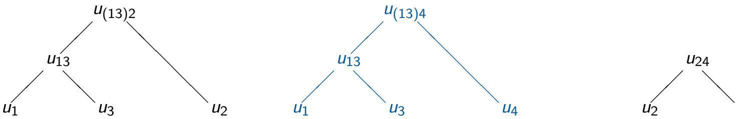

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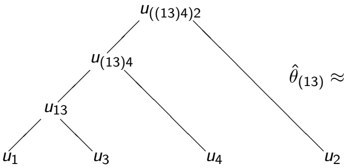

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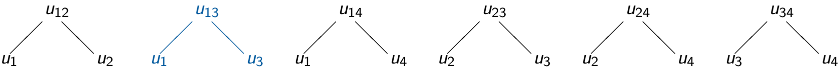
Estimation of HAC



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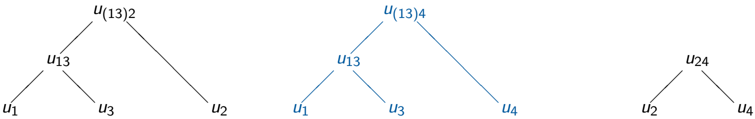

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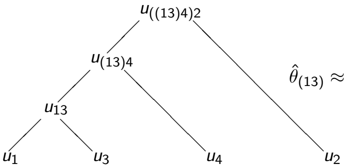

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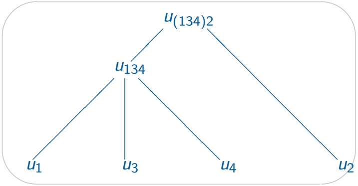
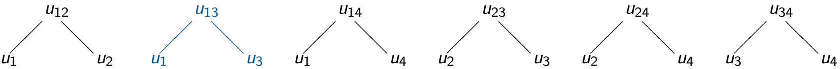


Penalized estimation of HAC



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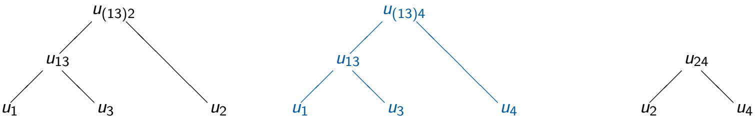

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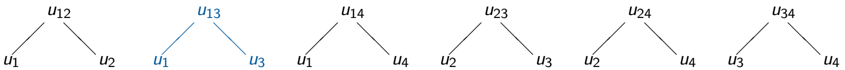

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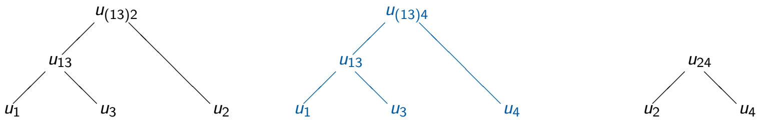
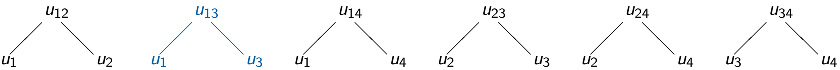
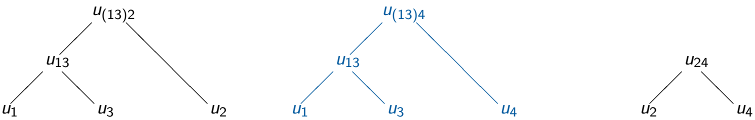


$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4}, \quad \text{if } \hat{\theta}_{13} - \hat{\theta}_{(13)4} < \epsilon_n \quad \Rightarrow$$

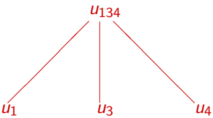
Penalized estimation of HAC



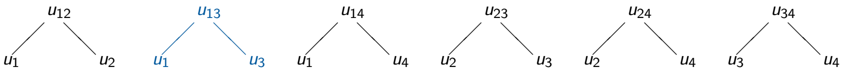
$$\max\{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{34}\} = \hat{\theta}_{13} \quad \Rightarrow$$



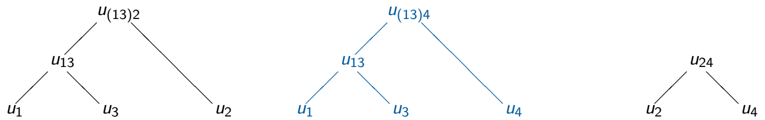
$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4}, \quad \text{if } \hat{\theta}_{13} - \hat{\theta}_{(13)4} < \epsilon_n \quad \Rightarrow$$



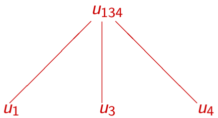
Penalized estimation of HAC



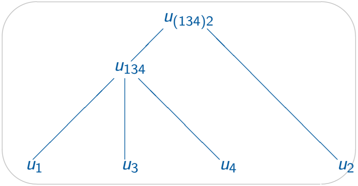
$$\max\{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{34}\} = \hat{\theta}_{13} \quad \Rightarrow$$

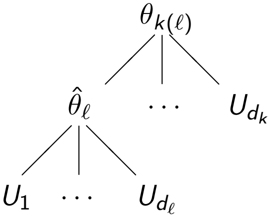


$$\max\{\hat{\theta}_{(13)2}, \hat{\theta}_{(13)4}, \hat{\theta}_{24}\} = \hat{\theta}_{(13)4}, \quad \text{if } \hat{\theta}_{13} - \hat{\theta}_{(13)4} < \epsilon_n \quad \Rightarrow$$



$\Longrightarrow$





- Build $\ell_i(\theta_{k(\ell)}) = \log\{c(U_{i1}, \ldots, U_{id_k}; \theta_{k(\ell)})\}$. ▸ Assumptions
- Penalized log-likelihood

$$\mathcal{Q}(\theta_\ell, \theta_{k(\ell)}) = \sum_{i=1}^{n} \ell_i(\theta_{k(\ell)}) - np_{\lambda_n}(\theta_\ell - \theta_{k(\ell)}),$$

  c.f. Cai and Wang (2014, JASA), Fan and Li (2001, JASA), Tibshirani et al. (2005, JRSSB).
- Let $\hat{\theta}^{\lambda_n}_{k(\ell)}$ be the maximizer of $\mathcal{Q}(\hat{\theta}_\ell, \theta_{k(\ell)})$.

Sparsity and oracle property

Proposition

Under Assumptions 1-3, if $n^{1/2}\lambda_n \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathrm{P}(\hat{\theta}^{\lambda_n}_{k(\ell)} = \theta_{\ell,0}) = 1.$$

Proposition

Under Assumptions 1-3, if $\lambda_n \to 0$ as $n \to \infty$, then

$$n^{1/2}\{\widehat{\mathcal{I}}(\theta_{k(\ell),0}) + p''_{\lambda_n}(\theta_0^-)\}\big[(\hat{\theta}^{\lambda_n}_{k(\ell)} - \theta_{k(\ell),0}) - \big\{\widehat{\mathcal{I}}(\theta_{k(\ell),0}) + p''_{\lambda_n}(\theta_0^-)\big\}^{-1} p'_{\lambda_n}(\theta_0^-)\big] \xrightarrow{\mathcal{L}} \mathrm{N}\{0, \mathcal{I}(\theta_{k(\ell),0})\},$$

where $\theta_0^- = \theta_{\ell,0} - \theta_{k(\ell),0}$.

ML representation

- Let $\hat{\theta}_{k(\ell)}$ and $\hat{\theta}_{\ell}$ be the MLE of Okhrin et al. (2013, JoE).
- Linear approximation of penalty function, Zou and Li (2008, Ann.).

Proposition

Under Assumptions 1-3, $\hat{\theta}_{k(\ell)}^{\lambda_n} = \hat{\theta}_{k(\ell)} + \epsilon_n$, *with*

$$\epsilon_n \stackrel{\text{def}}{=} \epsilon(\lambda_n, a_n) = \widehat{\mathcal{I}}(\hat{\theta}_{k(\ell)})^{-1} p'_{\lambda_n}(\hat{\theta}_{\ell} - \hat{\theta}_{k(\ell)}).$$

Practical issues

- Attain sparsity from

$$\hat{\theta}_{k(\ell)} = \hat{\theta}_{\ell}, \quad \text{if} \quad \hat{\theta}_{\ell} - \hat{\theta}_{k(\ell)} \leq \epsilon_n.$$

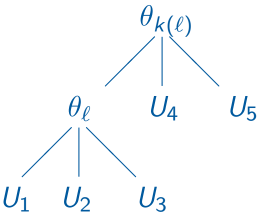
- Wang et al. (2007, Biometrica), determine $(\lambda, a)^\top$ by

$$(\lambda_n, a_n)^\top = \arg \max_{(\lambda,a)^\top} 2 \sum_{i=1}^{n} \ell_i \left\{ \hat{\theta}_{k(\ell)} + \epsilon(\lambda, a) \right\} - q_k \log(n).$$

- q_k parameters in HAC up to level k.

Setup

- Until $m = 1000$ structures correctly specified.
- Sample size $n = 250$.
- Let $\tau : \Theta_{k(\ell)} \to [0,1]$ transform the parameter $\theta_{k(\ell)}$ into Kendall's correlation coefficient.



- $\theta_\ell = \tau^{-1}(0.7)$ and $\theta_{k(\ell)} = \tau^{-1}(0.3)$.

Family	$s(\hat{\theta}) = s(\theta_0)$	$\tau(\hat{\theta}_1)$ (sd)	$\tau(\hat{\theta}_2)$ (sd)	$\#\{\hat{\theta}\}$
Clayton	0.82	0.70 (0.01)	0.30 (0.02)	3.04
Frank	0.85	0.70 (0.01)	0.30 (0.02)	3.03
Gumbel	0.85	0.70 (0.01)	0.30 (0.02)	3.02
Joe	0.88	0.70 (0.01)	0.30 (0.02)	3.04

Table 1: $s(\hat{\theta}) = s(\theta_0)$ reports the fraction of correctly specified structures, $\tau(\hat{\theta}_k)$ (sd), $k = 1, 2$, refers to the sample average of Kendall's $\tau(\cdot)$ evaluated at the estimates and sd to the sample standard deviation thereof. If the structure is misspecified, $\#\{\hat{\theta}\}$ gives the number of parameters on average included in the misspecified HAC.

Estimation strategy

- log-returns of ten stock indices are modeled by

$$\begin{aligned} X_t &= \mu_i\left(X_{t-1},\ldots\right) + \sigma_t\left(X_{t-1},\ldots\right)\varepsilon_t, \\ \varepsilon_t|\mathcal{F}_{t-1} &\sim C\{F_{\varepsilon_1}(x_{t1}),\ldots,F_{\varepsilon_d}(x_{td});\theta_t\}. \end{aligned}$$

- Series $\{X_{tj}\}_{t=1}^{T}$, $j = 1,\ldots,d$, are modeled by ARMA-APARCH with skew-t marginal distributions $F_{\varepsilon_j}(\cdot;\chi_j,\nu_j)$.
- Clayton-based HAC $C(\cdot;\theta_t)$ depending on $\{\theta_t\}_{t=1}^{T}$.
- Rolling window for a fixed structure: Jan 01st, 2007 – Apr 30th, 2014.

Index	χ	ν	$Q_{15}(\varepsilon_i)$	$Q_{15}(\varepsilon_i^2)$	AD GoF
DJIA	0.85	6.22	0.85	0.76	0.08
HSI	0.92	8.24	0.26	0.32	0.28
KOSPI	0.87	7.28	0.49	0.17	0.44
N225	0.89	10.55	0.77	0.03	0.23
SSEC	0.91	4.55	0.10	0.16	0.21
STI	0.90	12.89	0.16	0.03	0.83
SX5E	0.91	7.94	0.85	0.20	0.66
TAIEX	0.86	5.67	0.02	0.58	0.15
XAO	0.84	16.88	0.86	0.96	0.69

Table 2: The skewness χ and shape ν parameter of the margins, p-values of the Ljung-Box tests, $Q_{15}(\cdot)$, for 15 lags and the Anderson-Darling goodness of fit test (AD GoF).

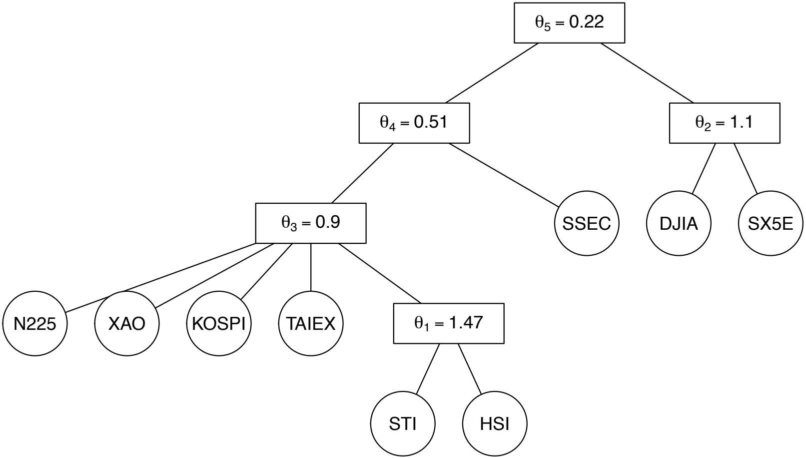


Figure 7: Sparsely estimated HAC for the entire data. ML estimation is implemented in R-package HAC, see Okhrin and Ristig (2014, JSS).

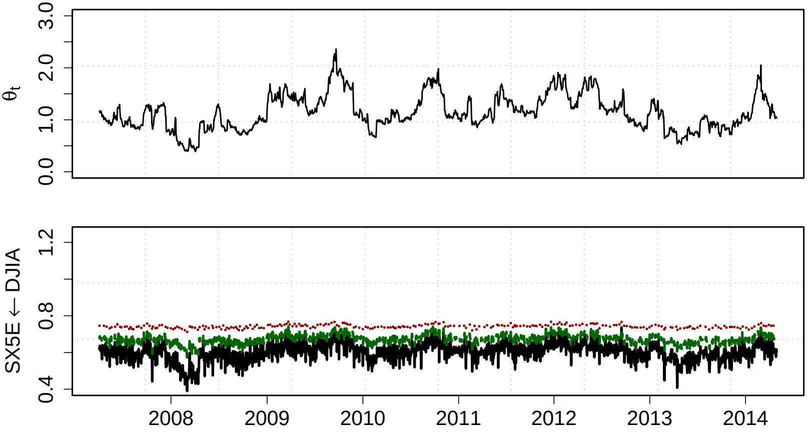


Figure 8: Upper panel shows estimates of $\hat{\theta}_{2,t}$ and lower panel the risk transmitted from DJIA to SX5E $\mathcal{S}^{\alpha}_{\text{SX5E}\leftarrow\text{DJIA}}$ for $\alpha \in \{0.1, 0.01, 0.0001\}$.

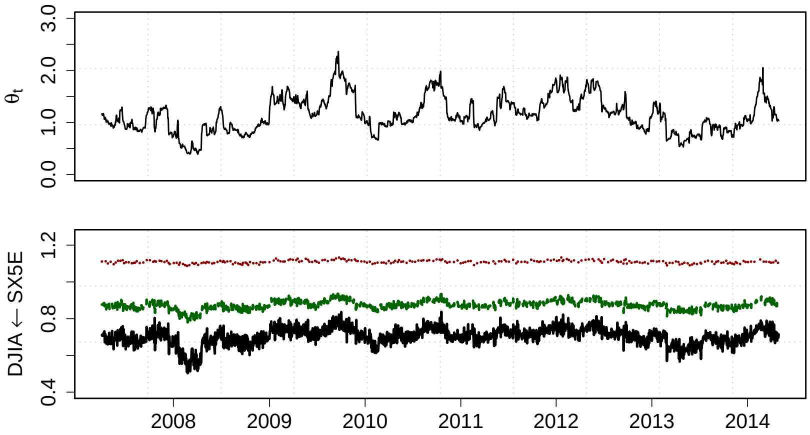


Figure 9: Upper panel shows estimates of $\hat{\theta}_{2,t}$ and lower panel the risk transmitted from SX5E to DJIA $\mathcal{S}^{\alpha}_{\text{DJIA}\leftarrow\text{SX5E}}$ for $\alpha \in \{0.1, 0.01, 0.0001\}$.

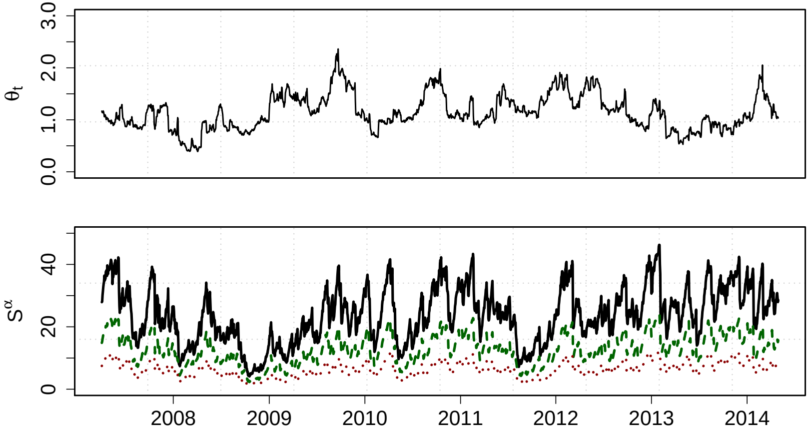


Figure 10: Upper panel shows estimates of $\hat{\theta}_{2,t}$ and lower panel systemic risk $\mathcal{S}^{\alpha}$ within the sub-portfolio SX5E and DJIA for $\alpha \in \{0.1, 0.01, 0.0001\}$.

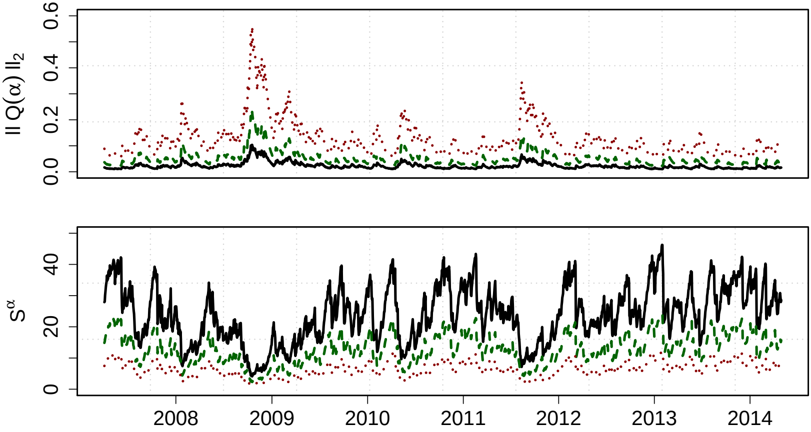


Figure 11: Upper panel shows $\|Q(\alpha)\|_2$, $Q(\alpha) = \{Q_{\text{DJIA}}(\alpha), Q_{\text{SX5E}}(\alpha)\}^\top$, and lower panel systemic risk $\mathcal{S}^\alpha$ within the sub-portfolio SX5E and DJIA for $\alpha \in \{0.1, 0.01, 0.0001\}$.

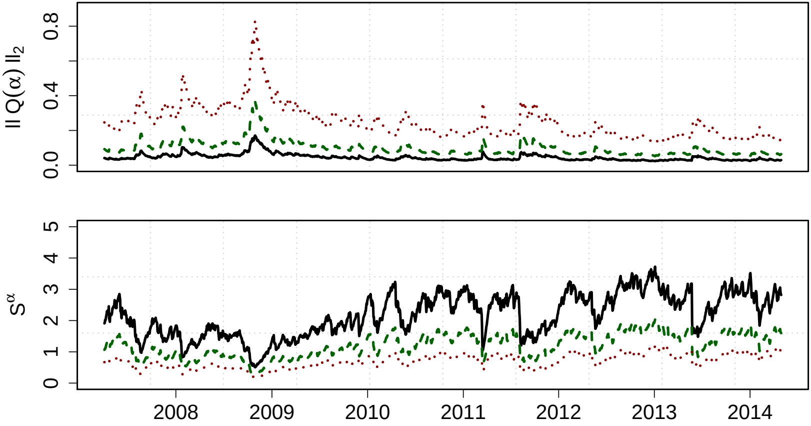


Figure 12: Upper panel shows $\|Q(\alpha)\|_2$ and lower panel systemic risk $\mathcal{S}^\alpha$ for the sub-portfolio HSI, KOSPI, N225, SSEC, STI, TAIEX and XAO, $\alpha \in \{0.1, 0.01, 0.0001\}$.

Conclusion

- Unified contagion and systemic measures based on conditional quantiles.
- Accuracy of the sparse HAC estimation is illustrated in a simulation study.
- Sparse estimation of HAC.
- Application reveals systemic risk due to contagion in tail area.

Conditional Systemic Risk with Penalized Copula

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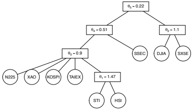
Alexander Ristig
Jeffrey Sheen
Stefan Trück



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Tail-monotonicity

Parzen (1979, JASA) calls a density function $h(x)$ with cdf $H(x)$ and tail exponent $\gamma > 0$ tail-monotone, if

- it is non-decreasing on an interval to the right of $a = \sup\{x : H(x) = 0\}$ and non-increasing on an interval to the left of $b = \inf\{x : H(x) = 1\}$, with $-\infty \leq a \leq b \leq \infty$;
- $h(x) > 0$ on $x \in (a, b)$ and $\sup_{x \in (a,b)} H(x)\{1 - H(x)\}|h'(x)|/h(x)^2 \leq \gamma$.
- Tail exponent $\gamma = \lim_{u \to 1}(u-1)\left[\log\left\{f\{Q(u)\}\right\}\right]'$

▸ Definitions

Assumptions

Define $\ell_i(\theta) = \log c(U_{i1}, \dots, U_{id_k}; \theta)$:

(1) Model is identifiable and $\theta_{k(\ell),0}$ is an interior point of the compact parameter space $\Theta_{k(\ell)}$. We assume that $\mathsf{E}_{\theta_{k(\ell)}}\{\ell_i'(\theta_{k(\ell)})\} = 0$ and information equality holds,

$$\mathcal{I}(\theta_{k(\ell)}) \stackrel{\text{def}}{=} \mathsf{E}_{\theta_{k(\ell)}}\left\{\ell_i'(\theta_{k(\ell)})^2\right\} = -\mathsf{E}_{\theta_{k(\ell)}}\left\{\ell_i''(\theta_{k(\ell)})\right\}$$

for $i = 1, \dots, n$.

(2) Fisher information $\mathcal{I}(\theta_{k(\ell)})$ is finite and strictly positive at $\theta_{k(\ell),0}$.

(3) There exists an open subset Ω of $\Theta_{k(\ell)}$ containing the true parameter $\theta_{k(\ell),0}$ such that for almost all U_i, $i = 1, \ldots, n$, the density $c(U_{i1}, \ldots, U_{id_k}; \theta_{k(\ell)})$ admits all third derivatives $c'''(\cdot; \theta_{k(\ell)})$ for all $\theta_{k(\ell)} \in \Omega$. Furthermore, there exist functions $M(\cdot)$ such that $\left|\ell_i'''(\theta_{k(\ell)})\right| \leq M(U_i)$, for all $\theta_{k(\ell)} \in \Omega$, with $\mathsf{E}\left\{M(U_i)\right\} < \infty$.

▸ Penalized ML

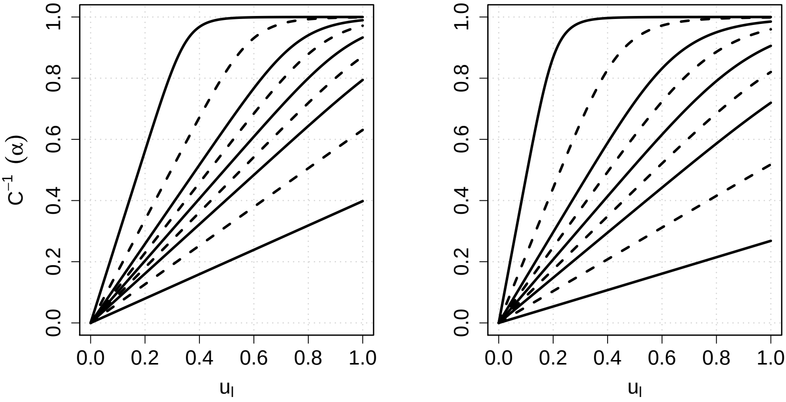

Figure 13: $C^{-1}_{U_{\boldsymbol{k}}|U_\ell=u_\ell}(\alpha)$ for Clayton copula. Alternating lines (solid and dashed) refer to $\alpha \in \{0.0001, 0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.9999\}$ – bottom-up ordered. Left panel illustrates $\theta = 9$ and right panel $\theta = 6$.

▸ Heterogenous margins

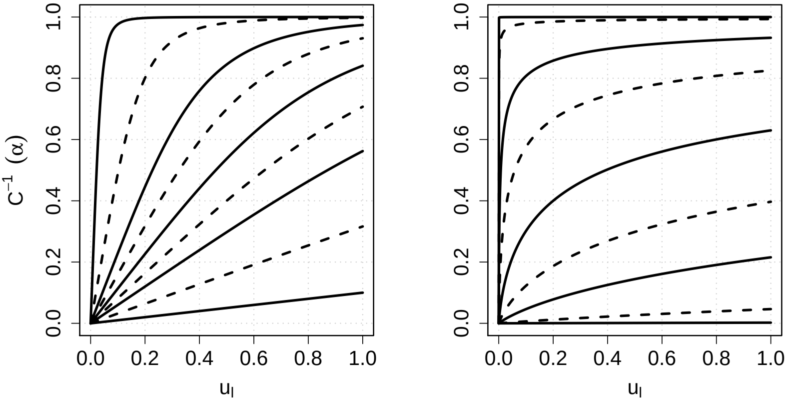

Figure 14: $C^{-1}_{U_{\boldsymbol{k}}|U_\ell=u_\ell}(\alpha)$ for Clayton copula. Alternating lines (solid and dashed) refer to $\alpha \in \{0.0001, 0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.9999\}$ – bottom-up ordered. Left panel illustrates $\theta = 3$ and right panel $\theta = 0.5$.

Heterogenous margins