TERES - Tail Event Risk Expected Shortfall

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Motivation

Risk Management

- Challenges
  - Expected shortfall ES_α - coherent; VaR_α - not coherent
  - Extreme value theory discards data
  - Historical estimation not feasible for small samples

Example: credit rating, $VaR_{0.0002}$, $ES_{0.001}$, $ES_{0.01}$

Coherence

Objectives

(i) Expected Shortfall (ES)

- M-quantiles: expectiles, quantiles
- Tail heaviness

(ii) TERES

- ES estimation: robustness; pseudo maximum likelihood
- Tail scenarios and ES range: risk level, lengthening the tail

Example 1

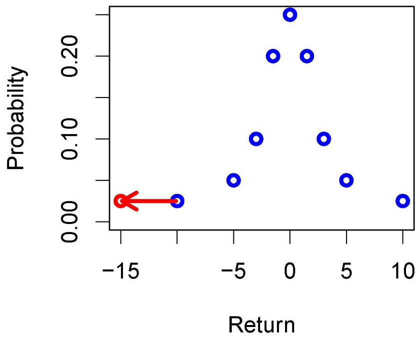

Figure 1: Discrete distribution of returns, $VaR_{0.05}$ remains unchanged if tail structure changes

Example 2

Expected Shortfall (lengthening the tail)

An investor holds a portfolio and investigates the theoretical ES at 1% level across two scenarios

Result

(a) Standard normal, $VaR_{0.01} = -2.33$, $ES_{0.01} = -2.66$

(b) Standard Laplace, $VaR_{0.01} = -3.91$, $ES_{0.01} = -4.91$

Example 3

Expected Shortfall (lengthening the tail)

An investor has a long position in the S&P 500 index and estimates ES at 1% level, 20000911-20140911 (3654 days)

TERES - standardized returns

(a) Standard normal

(b) Standard Laplace

Example 3

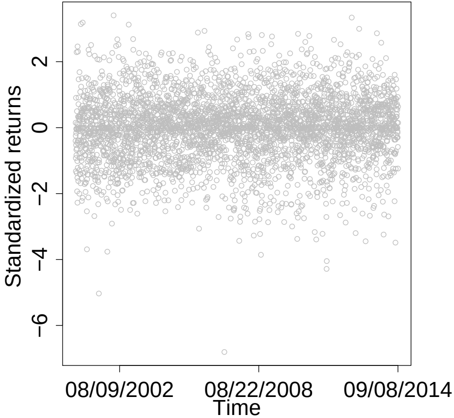

Figure 2: S&P 500 returns from 20000911-20140911 (3654 days)

Example 3

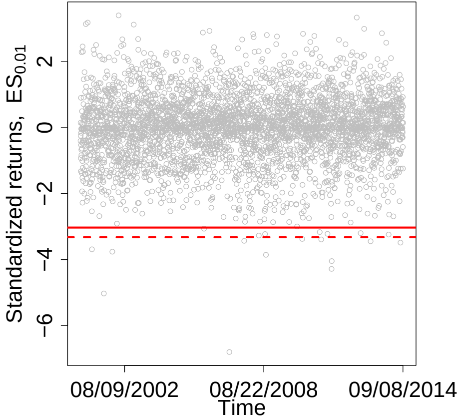

Figure 3: Estimated $ES_{0.01}$ using TERES, (a) standard normal - solid, (b) standard Laplace - dashed

Research Questions

How are M-Quantiles used for ES estimation?

How does the risk level α influence the variability of ES estimates?

Which range of ES is expected under different tail scenarios?

Outline

1. Motivation ✓
2. Expected Shortfall
3. TERES
4. Empirical Results
5. Conclusions

Expected Shortfall

- Standardized (portfolio) return Y with pdf $f(\cdot)$ and cdf $F(\cdot)$
- Expected shortfall

$$ES_\alpha = \mathsf{E}[Y | Y < q_\alpha]$$

with quantile $VaR_\alpha = q_\alpha = F^{-1}(\alpha)$ at risk level $\alpha \in [0, 1]$

M-Quantiles

- Loss function $\rho_{\alpha,\gamma}(u) = |\alpha - \mathbf{I}\{u < 0\}|\,|u|^{\gamma}$
  - Quantile - ALD location estimate
    $q_{\alpha} = \arg\min_{\theta} \mathsf{E}\,\rho_{\alpha,1}(Y - \theta)$
  - Expectile - AND location estimate
    $e_{\alpha} = \arg\min_{\theta} \mathsf{E}\,\rho_{\alpha,2}(Y - \theta)$

Loss Function

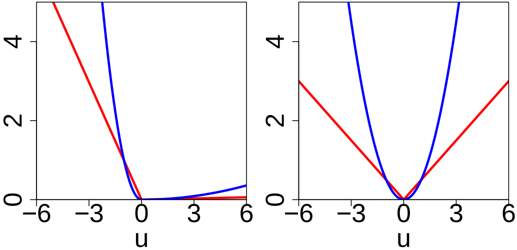

Figure 4: Expectile and quantile loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right)

LQRcheck

Tail Structure

- M-Quantiles
  - Level α, e_α and q_α
  - Level τ_α, $e_{\tau_\alpha} = q_\alpha$
- Taylor (2008) ▸ Proof

$$ES_\alpha = e_{\tau_\alpha} + \frac{e_{\tau_\alpha} - \mathsf{E}[Y]}{1 - 2\tau_\alpha} \frac{\tau_\alpha}{\alpha}$$

Expectiles and Quantiles

- Jones (1993), Guo and Härdle (2011) ▸ Proofs

$$\tau_\alpha = \frac{LPM_Y(q_\alpha) - q_\alpha \alpha}{2\{LPM_Y(q_\alpha) - q_\alpha \alpha\} + q_\alpha - \mathsf{E}[Y]}$$

$$LPM_Y(u) = \int_{-\infty}^{u} sf(s)ds$$

Example: $LPM_Y(q_\alpha) = -\varphi(q_\alpha)$ for $\mathsf{N}(0,1)$

TERES

- Flexible statistical framework - tail scenarios
- ES estimation
  1. Mixture distribution for Y or
  2. Loss function reparameterization - asymmetric generalized error distribution (GED)

Mixture Distribution

- Contamination level $\delta \in [0,1]$, Huber (1964)

$$F_\delta(x) = (1-\delta)\,\Phi(x) + \delta H(x)$$

with $H(\cdot)$ - cdf of a symmetrically distributed r.v., e.g., standard Laplace

Mixture Distribution

- Lengthening the tail
- Special cases
  - Standard normal, $\delta = 0$
  - Standard Laplace, $\delta = 1$

Expected Shortfall

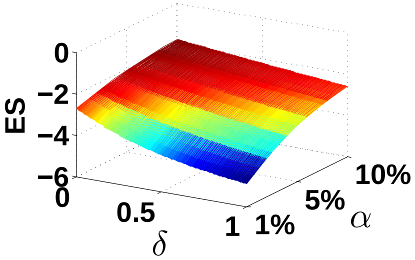

Figure 5: Theoretical ES assuming different contamination (δ) and risk levels (α)

Data

- Datastream: S&P 500 Index
- Span: 20000911-20140911 (3654 trading days)
- Standardized daily returns

Data

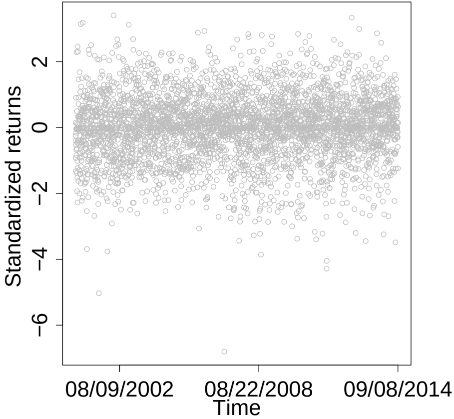

Figure 6: S&P 500 standardized returns

Expected Shortfall

- Risk level α: 0.01, 0.05 and 0.10
- Sample quantiles $\widehat{q}_\alpha$: -2.62, -1.43 and -1.03
- Contamination level

$$\delta \in \{0, 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.10, 0.15, 0.25, 0.5, 1\}$$

GARCH scaling

Expected Shortfall

δ	$ES_{0.10}$
0.0	-1.46
0.001	-1.46
0.002	-1.46
0.005	-1.46
0.01	-1.47
0.02	-1.47

δ	$ES_{0.10}$
0.05	-1.49
0.10	-1.51
0.15	-1.53
0.25	-1.58
0.50	-1.66
1.00	-1.73

Table 1: *ES* for the S&P 500 at $\alpha = 0.10$

Expected Shortfall

δ	$ES_{0.05}$	δ	$ES_{0.05}$
0.0	-1.86	0.05	-1.90
0.001	-1.86	0.10	-1.94
0.002	-1.86	0.15	-1.98
0.005	-1.87	0.25	-2.04
0.01	-1.87	0.50	-2.13
0.02	-1.88	1.00	-2.13

Table 2: *ES* for the S&P 500 at $\alpha = 0.05$

Expected Shortfall

δ	$ES_{0.01}$
0.0	-3.03
0.001	-3.03
0.002	-3.04
0.005	-3.05
0.01	-3.06
0.02	-3.09

δ	$ES_{0.01}$
0.05	-3.18
0.10	-3.28
0.15	-3.37
0.25	-3.45
0.50	-3.44
1.00	-3.32

Table 3: *ES* for the S&P 500 at $\alpha = 0.01$

Expected Shortfall

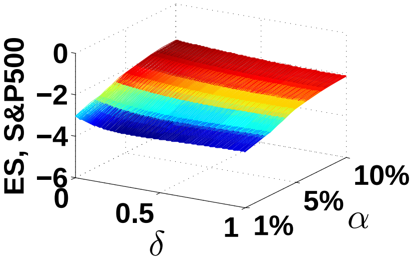

Figure 7: Expected shortfall using S&P 500 sample quantiles and assuming different contamination (δ) and risk levels (α).

Outlook

- δ-environment
  - Strict convexity
  - Analytical formula for Normal and Laplace cases
- Connection to Generalized Error Distribution (GED)
  - Risk level α is connected to skewness
  - Integration of moments into τ estimation

GED

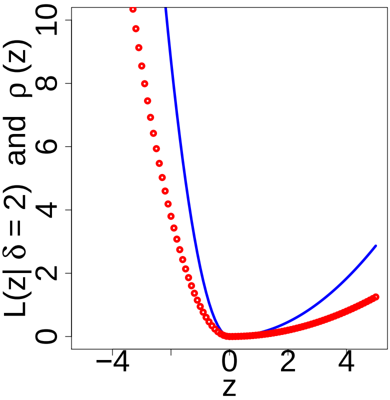

Figure 8: Asymmetric GED Likelihood and expectile loss function for $\alpha = 0.05$.

TERESGEDandMQuantile

Conclusions

(i) Expected Shortfall (ES)

- M-Quantiles applied successfully to estimate *ES*
- Interaction between α and τ illustrated

(ii) Estimating Expected Shortfall

- Distributional robustness: δ-neighborhood
- TERES: S&P 500 - $ES_{0.01}$, $ES_{0.05}$ and $ES_{0.10}$

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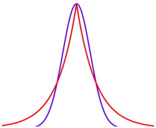

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Coherence

- Coherent risk measure $\rho(Y)$
  - Subadditivity, $\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2)$
  - Translation invariance, $\rho(Y + c) = \rho(Y)$ for constant c
  - Monotonicity, $\rho(Y_1) > \rho(Y_2) \quad \forall Y_1 < Y_2$
  - Positive homogeneity, $\rho(kY) = k\rho(Y) \quad \forall k > 0$

Risk Management

Subadditivity

- $\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2)$
- Diversification never increases risk
- Quantiles are not subadditive
- Expected shortfall is subadditive, Delbaen (1998)

▸ Risk Management

The expectile is defined as

$$e_{\tau_\alpha} = \arg\min_{\theta} \mathsf{E}\, \rho_{\tau_\alpha,2}\left(Y - \theta\right)$$

$$\rho_{\tau_\alpha,2}\left(u\right) = \left|\tau_\alpha - \mathbf{I}\left\{u < 0\right\}\right| \left|u\right|^2$$

For the continuous case

$$e_{\tau_\alpha} = \arg\min_{\theta} \int \rho_{\tau_\alpha,2}(Y - \theta)$$

This is a Quadratic convex problem with F.O.C.

$$(1 - \tau_\alpha) \int_{-\infty}^{s} (y - s) f(y) dy + \tau_\alpha \int_{s}^{\infty} (y - s) f(y) dy = 0$$

▸ Tail Structure

$$(1-\tau_\alpha)\int_{-\infty}^{e_{\tau_\alpha}}(y-e_{\tau_\alpha})f(y)dy+(1-\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$
$$=-\tau_\alpha\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy+(1-\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$

$$(1-\tau)\{\mathsf{E}(Y)-e_{\tau_\alpha}\}=(1-2\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$
$$e_{\tau_\alpha}-\mathsf{E}(Y)=\frac{(2\tau_\alpha-1)}{1-\tau_\alpha}\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$

This result is equal to (2.7) in Newey and Powell (1987)

▸ Tail Structure

Finally, as pointed out in Taylor (2008)

$$e_{\tau_\alpha} - \mathsf{E}[Y] = \frac{1-2\tau_\alpha}{\tau_\alpha}\,\mathsf{E}\left[(Y - e_{\tau_\alpha})\,\mathsf{I}\{Y > e_{\tau_\alpha}\}\right]$$

$$\mathsf{E}[Y|Y > e_{\tau_\alpha}] = e_{\tau_\alpha} + \frac{\tau(e_{\tau_\alpha} - \mathsf{E}[Y])}{(1-2\tau_\alpha)F(e_{\tau_\alpha})}$$

And using $e_{\tau_\alpha} = q_\alpha$

$$\mathsf{E}[Y|Y > q_\alpha] = e_{\tau_\alpha} + \frac{(e_{\tau_\alpha} - \mathsf{E}[Y])\tau_\alpha}{(1-2\tau_\alpha)\alpha}$$

$$= \mathsf{ES}(e_{\tau_\alpha}, \tau_\alpha|\alpha)$$

▸ Tail Structure

Relation of Expectiles and Quantiles

F.O.C. of Expectiles:

$$0 = (1-\tau_\alpha)\int_{-\infty}^{e_{\tau_\alpha}} (y - e_{\tau_\alpha}) f(y) dy + \tau_\alpha \int_{e_{\tau_\alpha}}^{\infty} (y - e_{\tau_\alpha}) f(y) dy$$

Reformulation yields

$$\tau_\alpha \left(e_{\tau_\alpha} - 2\int_{-\infty}^{e_{\tau_\alpha}} e_{\tau_\alpha} f(y) dy \right) + \int_{-\infty}^{e_{\tau_\alpha}} e_{\tau_\alpha} f(y) dy$$

$$= \tau_\alpha \left(\int_{-\infty}^{\infty} y f(y) dy - 2\int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy \right) + \int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy$$

▸ Expectiles and Quantiles

$$\tau_\alpha \left\{ 2 \left(\int_{-\infty}^{e_{\tau\alpha}} y f(y) dy - e_{\tau_\alpha} \int_{-\infty}^{e_{\tau\alpha}} f(y) dy \right) + e_{\tau_\alpha} - \mathsf{E}[Y] \right\}$$
$$= \int_{-\infty}^{e_{\tau\alpha}} y f(y) dy - \int_{-\infty}^{e_{\tau\alpha}} e_{\tau_\alpha} f(y) dy$$

And finally

$$\tau_\alpha = \frac{\mathrm{LPM}_{e_{\tau_\alpha}}(y) - e_{\tau_\alpha} F(e_{\tau_\alpha})}{2\left\{\mathrm{LPM}_{e_{\tau_\alpha}}(y) - e_{\tau_\alpha} F(e_{\tau_\alpha})\right\} + e_{\tau_\alpha} - \mathsf{E}[Y]}$$

▸ Expectiles and Quantiles

Tail Event Risk

Figure 9: $\alpha\tau(\alpha)$ for F_δ

▸ Expectiles and Quantiles

Standardization

- $\widehat{\sigma}_i$ from GARCH(1,1)

$$
\begin{aligned}
y_i =& \beta_0 + \beta_1 y_{i-1} + \varepsilon_i \\
\sigma_i^2 =& \alpha_0 + \alpha_1 \varepsilon_{i-1}^2 + \alpha_2 \sigma_{i-1}^2
\end{aligned}
$$

- $\widehat{e}_{0.5}$ is assumed time constant
- $\widehat{Y}_i = \frac{r_i - \widehat{e}_{0.5}}{\widehat{\sigma}_i}$

Back

Generalized Error Distribution

- Let $\kappa > 0$ and $g(x)$ be a symmetric distribution
- An asymmetric distribution $f(x)$ can be obtained as:

$$f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} g(x\kappa) & , 0 \leq x \\ g(\frac{x}{\kappa}) & , \text{ else} \end{cases} \tag{1}$$

- The Generalized Error Distribution (GED, Exponential Power distr.) is defined as

$$g(x|\gamma,\sigma,\theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})} \exp\left\{ - \left| \frac{x-\theta}{\sigma} \right|^{\gamma} \right\} \tag{2}$$

▸ Outlook

Following Ayebo and Kozubowski (2003), (1) and (2) yield a skew GED:

$$f(x|\gamma,\kappa,\sigma,\theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})}\frac{\kappa}{1+\kappa^2}\exp\left\{-\frac{\kappa^\gamma}{\sigma^\gamma}|x-\theta|_+^\gamma - \frac{1}{\kappa^\gamma\sigma^\gamma}|x-\theta|_-^\gamma\right\}$$

- Parameter
  - γ Shape, $\gamma = 1$ Laplace, $\gamma = 2$ Normal
  - κ Skewness, $\kappa = 1$ is symmetric
  - σ Scale
  - θ Mean

Outlook

- Part of $-\ln\{f(\cdot)\}$ that depends on x

$$\frac{\kappa^\gamma}{2\sigma^\gamma}\,|x-\theta|^\gamma\,\mathbf{I}\{x-\theta\le 0\}+\frac{1}{2\kappa^\gamma\sigma^\gamma}\,|x-\theta|^\gamma\,\mathbf{I}\{x-\theta<0\}$$

- M-quantile loss function

$$\begin{aligned}\rho(x-\theta) &= |\tau-\mathbf{I}\{x-\theta<0\}||x-\theta|^\gamma\\ &= \tau|x-\theta|^\gamma\mathbf{I}\{x-\theta\le 0\}+(1-\tau)|x-\theta|^\gamma\mathbf{I}\{x-\theta<0\}\end{aligned}$$

- M-Quantile-GED relation: $\frac{\alpha}{1-\alpha}\propto\frac{\kappa^\gamma}{\kappa^{-\gamma}}=\kappa^{2\gamma}$

▸ Outlook